

American and Bermudan options in currency markets under proportional transaction costs

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Abstract

The pricing and hedging of a general class of options (including American, Bermudan and European options) on multiple assets are studied in the context of currency markets where trading in all assets is subject to proportional transaction costs, and where the existence of a risk-free numéraire is not assumed. Probabilistic dual representations are obtained for the bid and ask prices of such options, together with constructions of hedging strategies, optimal stopping times and approximate martingale representations for both long and short option positions.

Keywords American options · Bermudan options · transaction costs · currency markets · superhedging · randomised stopping times · exercise policy

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1 Introduction

In this paper we consider the pricing and hedging of a wide class of options in Kabanov's model [8] of foreign exchange markets, where proportional transaction costs are modelled as bid-ask spreads between currencies. This model has been well studied; see e.g. [9, 10, 23].

The results of this paper apply to any option that can be described in full by a payoff process together with an exercise policy specifying the circumstances in which it can be exercised at each date up to its expiration. The class of such options is large as it contains American, Bermudan and European options. For such options, we compute the ask price (seller's price, upper hedging price) as

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well as the bid price (buyer's price, lower hedging price), and derive probabilistic dual representations for these prices. We also construct optimal superhedging trading strategies for the buyer and the seller, together with optimal stopping times consistent with the exercise policy that realize the option prices.

It is well known in complete models without transaction costs that the best stopping time for the holder of an American or Bermudan option is also the most expensive stopping time for the seller to hedge against, and that hedging against this particular stopping time protects the seller against all other stopping times. Chalasani & Jha [4] observed that this is no longer the case for American options in the presence of proportional transaction costs: to hedge against all pure stopping times, the seller must in effect be protected against a certain *randomised* stopping time (see Definition 2.5 below). Thus the optimal stopping times of the buyer and seller of an American option no longer coincide, and it may cost the seller more to hedge against all stopping times than to hedge against the best stopping time for the buyer. We will show below that this is true in general for any option that allows more than one exercise time (i.e. any non-European option).

There is a geometrical explanation for this apparent lack of symmetry. For both parties to an option, the price, optimal stopping time and optimal superhedging strategy solve a linear optimization problem over the set of superhedging strategies. The superhedging strategies for the seller form a convex set. In contrast, each superhedging strategy for the buyer hedges against a specific stopping time, so that a convex combination of two superhedging strategies for different stopping times may no longer be a superhedging strategy for the buyer. Thus the pricing problem (4.2) for the seller is convex, whereas if the exercise policy allows more than one stopping time, then the pricing problem (4.8) for the buyer is a mixed integer programming problem that is generally not convex (not even in the friction-free case; for American options see [15]).

The linear optimization problems (4.2) and (4.8) both grow exponentially with the number of time steps, even for options with path-independent payoffs (see [5, 22] for results on European options). Various special cases of European and American options have been studied in binomial two-asset models with proportional transaction costs. The replication of European options has been well studied [1, 3, 14, 16], and the first algorithm (with exponential running time) for computing the bid and ask prices for European options was established by Edirisinghe, Naik & Uppal [7]. In a similar technical setting, Kociński [11, 12] studied the exact replication of American options, Perrakis & Lefoll [17, 18] investigated the pricing of American call and put options, and Tokarz & Zastawniak [24] worked with general American options under small proportional transaction costs. Recently, Löhne & Rudloff [13] established an efficient algorithm for finding the set of superhedging strategies for European options in a similar technical setting to the present paper.

The main contribution of this paper is to establish martingale representations for the bid and ask prices of options on multiple assets. In doing so we extend the work of Chalasani & Jha [4] for American options with cash settlement in single-stock models. For American options in currency markets, we also

partially extend the results of Bouchard & Temam [2], who established the existence of a dual representation for the set of initial endowments that allows one to superhedge the seller's position, but did not provide a constructive proof and did not explore the case of the buyer. These constructions as well as the buyer's case are covered in the present paper. In addition, the construction of hedging strategies extends the work of Löhne & Rudloff [13] from European options to general derivatives (including European and American options). Moreover, in our results we are able to relax Schachermayer's robust no-arbitrage condition [23], which was assumed in [13], and require just the weak no-arbitrage property (2.4) due to Kabanov and Stricker [10].

The proof of the main results (Theorems 4.4 and 4.9) includes constructions of the sets of superhedging strategies and stopping times for both the buyer and seller, together with the approximate martingales and pricing measures involved in the martingale representations of both the bid and ask price of an option with general exercise policy (subject to mild regularity conditions) on multiple assets under proportional transaction costs of any magnitude in a general discrete time setting. Such constructions extend and improve upon each of the various special cases mentioned above, as well as the results previously reported for European and American options in two-asset models [20, 21]. These constructions are efficient in that their running length grows only polynomially with the number of time steps when pricing options with path dependent payoffs and exercise policies in recombining tree models.

The paper is organised as follows. In Section 2 we fix the notation, specify the market model with transaction costs, and review the notions of support functions, randomised stopping times and approximate martingales. The notion of an exercise policy is introduced in Section 3. The main pricing and hedging results for the buyer and seller are presented in Section 4 as Theorems 4.4 and 4.9, and various special cases are discussed. Section 5 is devoted to the proof of Theorem 4.4 for the seller, while Theorem 4.9 is proved in Section 6. Appendix A gives the proof of a technical lemma used in the proof of Theorem 4.4.

2 Preliminaries and notation

2.1 Convex sets and functions

For any set $A \in \mathbb{R}^d$, define

$$\sigma_i(A) := \{x = (x^1, \dots, x^d) \in A : x^i = 1\},$$

and define the *cone generated by A* as

$$\text{cone } A := \{\lambda x : \lambda \geq 0, x \in A\}.$$

We say that a non-empty cone $C \in \mathbb{R}^d$ is *compactly i-generated* if $\sigma_i(C)$ is compact, non-empty and $C = \text{cone } \sigma_i(C)$.

Let \cdot denote the scalar product in Euclidean space. For any set $A \in \mathbb{R}^d$, denote by A^* the polar of $-A$, i.e.

$$A^* := \{y \in \mathbb{R}^d : y \cdot x \geq 0 \text{ for all } x \in A\}.$$

If A is a non-empty closed convex cone, then A^* is also a non-empty closed convex cone [19, Theorem 14.1].

The *effective domain* of any convex function $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty, -\infty\}$ is defined as

$$\text{dom } f := \{y \in \mathbb{R}^d : f(y) < \infty\}.$$

The *epigraph* of f is defined as

$$\text{epi } f := \{(y_0, y) \in \mathbb{R} \times \mathbb{R}^d : y_0 \geq f(y)\}.$$

The function f is called *proper* if $\text{epi } f \neq \emptyset$ and $f(y) > -\infty$ for all $y \in \mathbb{R}^d$.

Define the *convex hull* $\text{conv } A$ of any set $A \subseteq \mathbb{R}^d$ as the smallest convex set containing A . Define the *convex hull* of a finite collection $g_1, \dots, g_n : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$ of proper convex functions as the greatest convex function majorised by g_1, \dots, g_n , equivalently

$$\text{conv}\{g_1, \dots, g_n\}(x) := \inf \sum_{k=1}^n \alpha_k g_k(x_k)$$

for each $x \in \mathbb{R}^d$, where the infimum is taken over all $x_k \in \mathbb{R}^d$ and $\alpha_k \geq 0$ for $k = 1, \dots, n$ such that

$$\sum_{k=1}^n \alpha_k = 1, \quad \sum_{k=1}^n \alpha_k x_k = x.$$

Also note that

$$\text{dom conv}\{g_1, \dots, g_n\} = \text{conv} \left[\bigcup_{k=1}^n \text{dom } g_k \right].$$

The *closure* $\text{cl } f$ of a proper convex function $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$ is defined as the unique function whose epigraph is

$$\text{epi}(\text{cl } f) = \overline{\text{epi } f}. \quad (2.1)$$

If f is not proper, then $\text{cl } f$ is defined as the constant function $-\infty$. A proper convex function f is called *closed* if $f = \text{cl } f$, equivalently if $\text{epi } f$ is closed.

Define the *support function* $\delta_A^* : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$ of a non-empty convex set $A \subseteq \mathbb{R}^d$ as

$$\delta_A^*(x) := \sup\{x \cdot y : y \in A\}.$$

The function δ_A^* is convex, proper and positively homogeneous. If A is closed, then δ_A^* is closed [19, Theorem 13.2]. We shall often make use of the identity

$$\delta_{\mathbb{R}^d}^*(y) = \begin{cases} 0 & \text{if } y = 0, \\ \infty & \text{if } y \neq 0. \end{cases} \quad (2.2)$$

2.2 Proportional transaction costs in a currency market model

In this paper we consider a market model with d currencies and discrete trading dates $t = 0, 1, \dots, T$ on a finite probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration (\mathcal{F}_t) . The exchange rates between the currencies are represented as an adapted matrix-valued process $(\pi_t^{ij})_{i,j=1}^d$, where for any $t = 0, \dots, T$ and $i, j = 1, \dots, d$ the quantity $\pi_t^{ij} > 0$ is the amount in currency i that needs to be exchanged in order to receive one unit of currency j at time t .

We assume without loss of generality that \mathcal{F}_0 is trivial, that $\mathcal{F}_T = 2^\Omega$ and that $\mathbb{Q}(\omega) > 0$ for all $\omega \in \Omega$. Let Ω_t be the collection of atoms (called *nodes*) of \mathcal{F}_t at any time t . A node $\nu \in \Omega_{t+1}$ at time $t+1$ is called a *successor* of a node $\mu \in \Omega_t$ at time t if $\nu \subseteq \mu$. Denote the collection of successors of any node μ by $\text{succ } \mu$.

We write \mathcal{L}_t for the family of \mathcal{F}_t -measurable \mathbb{R}^d -valued random variables, where for convenience $\mathcal{L}_{-1} := \mathcal{L}_0$. Throughout this paper we shall implicitly and uniquely identify random variables in \mathcal{L}_t with functions on Ω_t .

Writing \mathcal{L}_t^+ for the family of non-negative random variables in \mathcal{L}_t , a portfolio $x = (x^1, \dots, x^d) \in \mathcal{L}_t$ is called *soluble* whenever it can be exchanged into a portfolio in \mathcal{L}_t^+ without additional investment, i.e. if there exist \mathcal{F}_t -measurable random variables $\beta^{ij} \geq 0$ for $i, j = 1, \dots, d$ such that

$$x^j + \sum_{i=1}^d \beta^{ij} - \sum_{i=1}^d \beta^{ji} \pi_t^{ji} \geq 0 \text{ for all } j. \quad (2.3)$$

Here β^{ij} represents the number of units of currency j obtained by exchanging currency i . Let \mathcal{K}_t be the convex cone in \mathcal{L}_t generated by the d^2 vectors of the form $e^i - e^j \pi_t^{ji}$, where e^1, \dots, e^d is the canonical basis for \mathbb{R}^d . The solvency condition (2.3) can now be written as

$$x \in \mathcal{S}_t := \mathcal{L}_t^+ - \mathcal{K}_t.$$

We shall refer to \mathcal{S}_t as the *solvency cone*. Observe that the cones \mathcal{K}_t and \mathcal{S}_t are both polyhedral and therefore closed.

A *self-financing strategy* $y = (y_t)$ is a predictable \mathbb{R}^d -valued process with initial value $y_0 \in \mathcal{L}_0 = \mathbb{R}^d$ such that

$$y_t - y_{t+1} \in \mathcal{S}_t \text{ for all } t < T.$$

Denote the set of all self-financing strategies by Φ .

The model with transaction costs is said to satisfy the *weak no-arbitrage property* (NA^w) of Kabanov and Stricker [10] if

$$\{y_T : y \in \Phi \text{ and } y_0 = 0\} \cap \mathcal{L}_T^+ = \{0\}. \quad (2.4)$$

This formulation is formally different but equivalent to that of Kabanov and Stricker [10], and was introduced by Schachermayer [23], who called it simply the no-arbitrage property.

We have the following fundamental result.

Theorem 2.1 ([10, 23]). The model satisfies the weak no-arbitrage property if and only if there exist a probability measure \mathbb{P} equivalent to \mathbb{Q} and an \mathbb{R}^d -valued \mathbb{P} -martingale $S = (S_t^1, \dots, S_t^d)$ such that

$$0 < S_t^j \leq \pi_t^{ij} S_t^i \text{ for all } i, j, t. \quad (2.5)$$

Remark 2.2. Condition (2.5) can equivalently be written as

$$S_t \in \mathcal{S}_t^* \setminus \{0\} \text{ for all } t.$$

If the model satisfies the weak no-arbitrage property, then \mathcal{S}_t^* is a non-empty polyhedral cone, and it is compactly i -generated with

$$\begin{aligned} \sigma_i(\mathcal{S}_t^{*\mu}) = \Big\{ (s^1, \dots, s^d) \in \mathbb{R}^d : s^i = 1, \frac{1}{\pi_t^{ji\mu}} \leq s^j \leq \pi_t^{ij\mu} \text{ for all } j \neq i, \\ s^j \leq \pi_t^{kj\mu} s^k \text{ for all } j \neq i, k \neq i \Big\} \end{aligned}$$

for all $\mu \in \Omega_t$.

Definition 2.3 (Equivalent martingale pair). A pair (\mathbb{P}, S) satisfying the conditions of Theorem 2.1 is called an *equivalent martingale pair*.

Denote the family of equivalent martingale pairs by \mathcal{P} . Let

$$\mathcal{P}^i := \{(\mathbb{P}, S) \in \mathcal{P} : S_t^i = 1 \text{ for all } t\}$$

for all $i = 1, \dots, d$.

We assume from here on that the model satisfies the weak no-arbitrage property, so that $\mathcal{P} \neq \emptyset$, equivalently $\mathcal{P}^i \neq \emptyset$ for all i .

Example 2.4. Consider three assets, where asset 3 is a cash account. Suppose that in a friction-free market assets 1 and 2 can be bought/sold, respectively, for $S^1 = 12$ and $S^2 = 8$ units of cash (asset 3). The friction-free exchange rate matrix would then be

$$\begin{bmatrix} 1 & S^2/S^1 & 1/S^1 \\ S^1/S^2 & 1 & 1/S^2 \\ S^1 & S^2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2/3 & 1/12 \\ 3/2 & 1 & 1/8 \\ 12 & 8 & 1 \end{bmatrix}.$$

Now assume that whenever an asset i is exchanged into a different asset j , transaction costs are charged at a fixed rate $k \geq 0$ against asset i , resulting in each off-diagonal exchange rate increased by a factor $1 + k$. If $k = \frac{1}{3}$, the exchange rate matrix becomes

$$\pi = \begin{bmatrix} 1 & (1+k)S^2/S^1 & (1+k)/S^1 \\ (1+k)S^1/S^2 & 1 & (1+k)/S^2 \\ (1+k)S^1 & (1+k)S^2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 8/9 & 1/9 \\ 1/4 & 1 & 1/6 \\ 16 & 32/3 & 1 \end{bmatrix}.$$

The cone \mathcal{S} consisting of solvent portfolios (x^1, x^2, x^3) and the section $\sigma_3(\mathcal{S}^*)$, which generates the cone \mathcal{S}^* , are shown in Figures 1(a), (b), respectively.

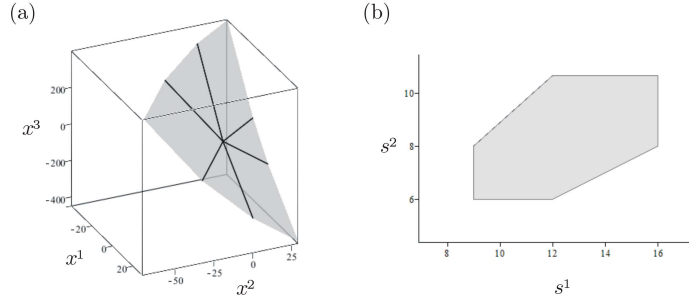


Figure 1: The solvency cone \mathcal{S} and section $\sigma_3(\mathcal{S}^*)$ of the cone \mathcal{S}^* in Example 2.4

2.3 Randomised stopping times

Definition 2.5 (Randomised stopping time). A *randomised (or mixed) stopping time* $\chi = (\chi_t)$ is a non-negative adapted process such that

$$\sum_{t=0}^T \chi_t = 1.$$

We write \mathcal{X} for the collection of all randomised stopping times.

Let \mathcal{T} be the set of (ordinary) stopping times. Any stopping time $\tau \in \mathcal{T}$ can be represented by the randomised stopping time $\chi^\tau = (\chi_t^\tau) \in \mathcal{X}$ defined by

$$\chi_t^\tau := \mathbf{1}_{\{\tau=t\}}$$

for all t . Here $\mathbf{1}_A$ denotes the indicator function of $A \subseteq \Omega$.

For any adapted process $X = (X_t)$ and $\chi \in \mathcal{X}$, define the *value of X at χ* as

$$X_\chi := \sum_{t=0}^T \chi_t X_t.$$

Moreover, define the processes $\chi^* = (\chi_t^*)$ and $X^{\chi^*} = (X_t^{\chi^*})$ as

$$\chi_t^* := \sum_{s=t}^T \chi_s, \quad X_t^{\chi^*} := \sum_{s=t}^T \chi_s X_s$$

for all t . Observe that χ^* is a predictable process since

$$\chi_t^* = 1 - \sum_{s=0}^{t-1} \chi_s$$

whenever $t > 0$. For notational convenience define

$$\chi_{T+1}^* := 0, \quad X_{T+1}^{\chi^*} := 0. \quad (2.6)$$

Definition 2.6 (χ -approximate martingale pair). For any $\chi \in \mathcal{X}$ the pair (\mathbb{P}, S) is called a χ -approximate martingale pair if \mathbb{P} is a probability measure and S an adapted process satisfying

$$S_t \in \mathcal{S}_t^* \setminus \{0\}, \quad \mathbb{E}_{\mathbb{P}}(S_{t+1}^{X^*} | \mathcal{F}_t) \in \mathcal{S}_t^*$$

for all t . If in addition \mathbb{P} is equivalent to \mathbb{Q} , then (\mathbb{P}, S) is called a χ -approximate equivalent martingale pair.

Denote the family of χ -approximate martingale pairs (\mathbb{P}, S) by $\bar{\mathcal{P}}(\chi)$, and write $\mathcal{P}(\chi)$ for the family of χ -approximate equivalent martingale pairs. For an ordinary stopping time $\tau \in \mathcal{T}$ we write $\mathcal{P}(\tau) := \mathcal{P}(\chi^\tau)$ and $\bar{\mathcal{P}}(\tau) := \bar{\mathcal{P}}(\chi^\tau)$ and say that (\mathbb{P}, S) is a τ -approximate (equivalent) martingale pair whenever it is a χ^τ -approximate (equivalent) martingale pair.

For any $\chi \in \mathcal{X}$ and $i = 1, \dots, d$ define

$$\begin{aligned} \bar{\mathcal{P}}^i(\chi) &:= \{(\mathbb{P}, S) \in \bar{\mathcal{P}}(\chi) : S_t^i = 1 \text{ for all } t\}, \\ \mathcal{P}^i(\chi) &:= \{(\mathbb{P}, S) \in \mathcal{P}(\chi) : S_t^i = 1 \text{ for all } t\}. \end{aligned}$$

Noting that $\mathcal{P} \subseteq \mathcal{P}(\chi) \subseteq \bar{\mathcal{P}}(\chi)$, it follows that $\mathcal{P}^i \subseteq \mathcal{P}^i(\chi) \subseteq \bar{\mathcal{P}}^i(\chi)$, and the weak no-arbitrage property implies that all these families are non-empty.

We have the following simple result.

Lemma 2.7. Fix any $i = 1, \dots, d$, and let ξ be any adapted \mathbb{R}^d -valued process. Then for any $\delta > 0$, any $\chi \in \mathcal{X}$ and any $(\mathbb{P}, \bar{S}) \in \bar{\mathcal{P}}^i(\chi)$ there exists a χ -approximate martingale pair $(\mathbb{P}^\delta, S^\delta) \in \mathcal{P}^i(\chi)$ such that

$$|\mathbb{E}_{\mathbb{P}^\delta}((\xi \cdot S^\delta)_\chi) - \mathbb{E}_{\bar{\mathbb{P}}}((\xi \cdot \bar{S})_\chi)| < \delta.$$

Proof. The weak no-arbitrage property guarantees the existence of some $(\mathbb{P}, S) \in \mathcal{P}^i \subseteq \mathcal{P}^i(\chi)$. If $\mathbb{E}_{\mathbb{P}}((\xi \cdot S)_\chi) = \mathbb{E}_{\bar{\mathbb{P}}}((\xi \cdot \bar{S})_\chi)$, then the claim holds with $\mathbb{P}^\delta := \mathbb{P}$ and $S^\delta := S$. If not, fix $\varepsilon := \min \{1, \frac{\delta}{2} / |\mathbb{E}_{\mathbb{P}}((\xi \cdot S)_\chi) - \mathbb{E}_{\bar{\mathbb{P}}}((\xi \cdot \bar{S})_\chi)|\}$, let $\mathbb{P}^\delta := (1 - \varepsilon)\bar{\mathbb{P}} + \varepsilon\mathbb{P}$ and

$$S_t^\delta := (1 - \varepsilon)\bar{S}_t \mathbb{E}_{\bar{\mathbb{P}}^\delta} \left(\frac{d\bar{\mathbb{P}}}{d\mathbb{P}^\delta} \middle| \mathcal{F}_t \right) + \varepsilon S_t \mathbb{E}_{\mathbb{P}^\delta} \left(\frac{d\mathbb{P}}{d\mathbb{P}^\delta} \middle| \mathcal{F}_t \right) \text{ for all } t.$$

□

3 Exercise policies

In the next section and onwards we will consider the pricing and hedging of an option that may only be exercised in certain situations, namely at any time t the owner of the option is only allowed to exercise on a subset \mathcal{E}_t of Ω . This setting contains a wide class of options, for example:

- A European option corresponds to $\mathcal{E}_T = \Omega$ and $\mathcal{E}_t = \emptyset$ for all $t < T$.

- A Bermudan option with exercise dates $t_1 < \dots < t_n$ corresponds to

$$\mathcal{E}_t = \begin{cases} \Omega & \text{if } t = t_1, \dots, t_n, \\ \emptyset & \text{otherwise.} \end{cases} \quad (3.1)$$

- An American option corresponds to $\mathcal{E}_t = \Omega$ for all t .
- An American-style option with random expiration date $\tau \in \mathcal{T}$ corresponds to $\mathcal{E}_t = \{\tau \geq t\}$ for all t .

The introduction of exercise policies allows the unification and extension of existing results for specific types of options, most notably European and American options. More immediately, we shall also use exercise policies as a theoretical tool in Section 6 when deriving the pricing and hedging theorem for the buyer from corresponding results for the seller of a related European-style option.

An exercise policy is formally defined as follows.

Definition 3.1 (Exercise policy). An *exercise policy* $\mathcal{E} \equiv (\mathcal{E}_t)$ is a sequence of subsets of Ω such that $\mathcal{E}_t \in \mathcal{F}_t$ for all t ,

$$\bigcup_{s=t+1}^T \mathcal{E}_s \in \mathcal{F}_t \text{ for all } t < T, \quad (3.2)$$

and

$$\bigcup_{t=0}^T \mathcal{E}_t = \Omega. \quad (3.3)$$

The condition $\mathcal{E}_t \in \mathcal{F}_t$ is consistent with the intuitive notion of allowing the buyer to make exercise decisions based on information available at time t . Condition (3.2) is consistent with allowing the buyer to determine on the basis of information currently available whether or not there are future opportunities for exercise. Condition (3.3) ensures that there is at least one opportunity to exercise the option in each scenario.

Define the sequence $\mathcal{E}^* = (\mathcal{E}_t^*)$ of sets associated with an exercise policy \mathcal{E} as

$$\mathcal{E}_t^* := \bigcup_{s=t}^T \mathcal{E}_s$$

for all t . For each t , the set \mathcal{E}_t^* contains those scenarios in which it is possible to exercise the option in at least one of the time steps t, \dots, T . Write $\mathcal{E}_{T+1}^* := \emptyset$ for convenience.

For an exercise policy $\mathcal{E} = (\mathcal{E}_t)$, define the sets of *randomised and ordinary stopping times consistent with \mathcal{E}* as

$$\begin{aligned} \mathcal{X}^{\mathcal{E}} &:= \{\chi \in \mathcal{X} : \{\chi_t > 0\} \subseteq \mathcal{E}_t \text{ for all } t\}, \\ \mathcal{T}^{\mathcal{E}} &:= \{\tau \in \mathcal{T} : \{\tau = t\} \subseteq \mathcal{E}_t \text{ for all } t\}. \end{aligned}$$

The following result specifies the relationship between \mathcal{E} , $\mathcal{T}^{\mathcal{E}}$ and $\mathcal{X}^{\mathcal{E}}$.

Proposition 3.2. For all t ,

$$\mathcal{E}_t = \bigcup_{\tau \in \mathcal{T}^\mathcal{E}} \{\tau = t\} = \bigcup_{\chi \in \mathcal{X}^\mathcal{E}} \{\chi_t > 0\}.$$

Proof. For the first equality, it is clear from the definition of $\mathcal{T}^\mathcal{E}$ that

$$\bigcup_{\tau \in \mathcal{T}^\mathcal{E}} \{\tau = t\} \subseteq \mathcal{E}_t \text{ for all } t.$$

We now show for any $t' = 0, \dots, T$ that there exists a stopping time $\tau' \in \mathcal{T}^\mathcal{E}$ such that $\{\tau' = t'\} = \mathcal{E}_{t'}$. Define

$$\mathcal{E}'_t := \begin{cases} \mathcal{E}_t \setminus \mathcal{E}_{t+1}^* & \text{if } t < t', \\ \mathcal{E}_{t'} & \text{if } t = t', \\ \mathcal{E}_t \setminus \bigcup_{s=0}^{t'-1} \mathcal{E}'_s & \text{if } t > t', \end{cases}$$

so that $\mathcal{E}'_0, \dots, \mathcal{E}'_T$ is a sequence of mutually disjoint sets in Ω with $\mathcal{E}'_t \subseteq \mathcal{E}_t$ and $\mathcal{E}'_t \in \mathcal{F}_t$ for all t . Moreover it is a partition of Ω since

$$\begin{aligned} \bigcup_{t=0}^T \mathcal{E}'_t &= \left\{ \bigcup_{t=0}^{t'-1} \mathcal{E}'_t \right\} \cup \mathcal{E}'_{t'} \cup \left\{ \mathcal{E}_{t'+1}^* \setminus \bigcup_{t=0}^{t'} \mathcal{E}'_t \right\} \\ &= \left\{ \bigcup_{t=0}^{t'-1} [\mathcal{E}_t \setminus \mathcal{E}_{t+1}^*] \right\} \cup \mathcal{E}_{t'}^* = \Omega. \end{aligned}$$

The random variable

$$\tau' := \sum_{t=0}^T t 1_{\mathcal{E}'_t}$$

is therefore a stopping time in $\mathcal{T}^\mathcal{E}$ with $\{\tau' = t'\} = \mathcal{E}'_{t'} = \mathcal{E}_{t'}$ as required.

The second equality holds because

$$\mathcal{E}_t = \bigcup_{\tau \in \mathcal{T}^\mathcal{E}} \{\tau = t\} \subseteq \bigcup_{\chi \in \mathcal{X}^\mathcal{E}} \{\chi_t > 0\} \subseteq \mathcal{E}_t$$

for all t . □

4 Main results and discussion

An *option* consists of an adapted \mathbb{R}^d -valued payoff process $\xi = (\xi_t)$ and an exercise policy \mathcal{E} . The seller delivers the portfolio $\xi_\tau \in \mathcal{L}_\tau$ to the buyer at a stopping time $\tau \in \mathcal{T}^\mathcal{E}$ chosen by the buyer among the stopping times consistent with \mathcal{E} .

4.1 Pricing and hedging for the seller

Consider the hedging and pricing problem for the seller of the option (ξ, \mathcal{E}) . A self-financing trading strategy $y \in \Phi$ is said to *superhedge* (ξ, \mathcal{E}) for the seller if

$$y_\tau - \xi_\tau \in \mathcal{S}_\tau \text{ for all } \tau \in \mathcal{T}^\mathcal{E}. \quad (4.1)$$

Definition 4.1 (Ask price). The *ask price* or *seller's price* or *upper hedging price* of (ξ, \mathcal{E}) at time 0 in terms of any currency $i = 1, \dots, d$ is defined as

$$\pi_i^a(\xi, \mathcal{E}) := \inf\{x \in \mathbb{R} : y \in \Phi \text{ with } y_0 = xe^i \text{ superhedges } (\xi, \mathcal{E}) \text{ for the seller}\}. \quad (4.2)$$

The interpretation of the ask price is that an endowment of at least $\pi_i^a(\xi, \mathcal{E})$ units of asset i at time 0 would enable an investor to settle the option without risk. A superhedging strategy y for the seller is called *optimal* if $y_0 = \pi_i^a(\xi, \mathcal{E})e^i$.

Our main aims are to compute and represent the option price $\pi_i^a(\xi, \mathcal{E})$, to construct the set of initial endowments that allow the seller to superhedge, and to construct an optimal superhedging strategy $y \in \Phi$ for the seller. To this end, consider the following construction.

Construction 4.2. For all t , let

$$\mathcal{U}_t^a := \begin{cases} \xi_t + \mathcal{S}_t & \text{on } \mathcal{E}_t, \\ \mathbb{R}^d & \text{on } \Omega \setminus \mathcal{E}_t. \end{cases} \quad (4.3)$$

Define

$$\mathcal{V}_T^a := \mathcal{W}_T^a := \mathcal{L}_T, \quad \mathcal{Z}_T^a := \mathcal{U}_T^a.$$

For $t < T$, let

$$\begin{aligned} \mathcal{W}_t^a &:= \mathcal{Z}_{t+1}^a \cap \mathcal{L}_t, \\ \mathcal{V}_t^a &:= \mathcal{W}_t^a + \mathcal{S}_t, \end{aligned} \quad (4.4)$$

$$\mathcal{Z}_t^a := \mathcal{U}_t^a \cap \mathcal{V}_t^a. \quad (4.5)$$

For each t the set \mathcal{U}_t^a is the collection of portfolios in \mathcal{L}_t that allows the seller to settle the option at time t . We shall demonstrate in Proposition 5.2 below that for each $t < T$ the sets \mathcal{V}_t^a , \mathcal{W}_t^a and \mathcal{Z}_t^a have natural interpretations as collections of portfolios that are of importance to the seller of the option. The set \mathcal{W}_t^a is the collection of portfolios at time t that allow the seller to settle the option in the future (at time $t + 1$ or later). The set \mathcal{V}_t^a consists of those portfolios that may be rebalanced at time t into a portfolio in \mathcal{W}_t^a , and \mathcal{Z}_t^a consists of all portfolios that allow the seller to settle the option at time t or any time in the future.

Remark 4.3. On \mathcal{E}_t , where exercise is allowed, the set \mathcal{U}_t^a is a translation of \mathcal{S}_t , so it is non-empty and polyhedral. It is then straightforward to show by backward induction that the following holds for all t :

- $\mathcal{V}_t^a, \mathcal{W}_t^a, \mathcal{Z}_t^a$ are all non-empty.
- $\mathcal{V}_t^a = \mathcal{W}_t^a = \mathcal{Z}_t^a = \mathbb{R}^d$ on $\Omega \setminus \mathcal{E}_t^*$.
- $\mathcal{Z}_t^a = \mathcal{V}_t^a$ on $\Omega \setminus \mathcal{E}_t$ and $\mathcal{Z}_t^a = \mathcal{U}_t^a$ on $\Omega \setminus \mathcal{E}_{t+1}^*$.
- \mathcal{V}_t^a and \mathcal{W}_t^a are polyhedral on \mathcal{E}_{t+1}^* and \mathcal{Z}_t^a is polyhedral on \mathcal{E}_t^* .

Note in particular that the non-empty set \mathcal{Z}_0^a is polyhedral since $\mathcal{E}_0^* = \Omega$.

The main pricing and hedging result for the seller reads as follows.

Theorem 4.4. The set \mathcal{Z}_0^a is the collection of initial endowments allowing the seller to superhedge (ξ, \mathcal{E}) , and

$$\begin{aligned}
\pi_i^a(\xi, \mathcal{E}) &= \max_{\chi \in \mathcal{X}^\mathcal{E}} \max_{(\mathbb{P}, S) \in \hat{\mathcal{P}}^i(\chi)} \mathbb{E}_{\mathbb{P}}((\xi \cdot S)_\chi) \\
&= \max_{\chi \in \mathcal{X}^\mathcal{E}} \sup_{(\mathbb{P}, S) \in \hat{\mathcal{P}}^i(\chi)} \mathbb{E}_{\mathbb{P}}((\xi \cdot S)_\chi) \\
&= \min\{x \in \mathbb{R} : xe^i \in \mathcal{Z}_0^a\} \\
&= -\min\{Z_0^a(s) : s \in \sigma_i(\mathbb{R}^d)\},
\end{aligned}$$

where Z_0^a is the support function of $-\mathcal{Z}_0^a$. An optimal superhedging strategy $y \in \Phi$ for the seller can be constructed algorithmically, and so can a randomised stopping time $\hat{\chi} \in \mathcal{X}^\mathcal{E}$ and $\hat{\chi}$ -approximate martingale pair $(\hat{\mathbb{P}}, \hat{S}) \in \hat{\mathcal{P}}^i(\hat{\chi})$ such that

$$\mathbb{E}_{\hat{\mathbb{P}}}((\xi \cdot \hat{S})_{\hat{\chi}}) = \pi_i^a(\xi, \mathcal{E}). \quad (4.6)$$

Any stopping time $\hat{\chi}$ and $\hat{\chi}$ -approximate martingale pair $(\hat{\mathbb{P}}, \hat{S}) \in \hat{\mathcal{P}}^i(\hat{\chi})$ satisfying (4.6) are called *optimal for the seller* of (ξ, \mathcal{E}) . Note that the optimal superhedging strategy, optimal stopping time, approximate martingale pair and strategy are not unique in general.

The proof of Theorem 4.4 appears in Section 5 below, together with details of the construction of the optimal superhedging strategy, stopping time and approximate martingale pair for the seller.

4.2 Pricing and hedging for the buyer

Consider now the pricing and hedging problem for the buyer of the option (ξ, \mathcal{E}) . A pair (y, τ) consisting of a self-financing trading strategy $y \in \Phi$ and a stopping time $\tau \in \mathcal{T}^\mathcal{E}$ *superhedges* (ξ, \mathcal{E}) for the buyer if

$$y_\tau + \xi_\tau \in \mathcal{S}_\tau. \quad (4.7)$$

Definition 4.5 (Bid price). The *bid price* or *buyer's price* or *lower hedging price* of (ξ, \mathcal{E}) at time 0 in terms of currency i is defined as

$$\pi_i^b(\xi, \mathcal{E}) := \sup\{-x \in \mathbb{R} : (y, \tau) \text{ with } y_0 = xe^i \text{ superhedges } (\xi, \mathcal{E}) \text{ for the buyer}\}. \quad (4.8)$$

The interpretation of the bid price is that $\pi_i^b(\xi, \mathcal{E})$ is the largest amount in currency i that can be raised at time 0 by the owner of (ξ, \mathcal{E}) by setting up a self-financing trading strategy with the property that it leaves him in a solvent position after exercising (ξ, \mathcal{E}) . A superhedging strategy (y, τ) for the buyer is called *optimal* if $y_0 = -\pi_i^b(\xi, \mathcal{E})e^i$.

Just as in the seller's case, the aims are to compute $\pi_i^b(\xi, \mathcal{E})$, to establish a probabilistic representation for it, to find the set of initial endowments allowing superhedging for the buyer, and to construct an optimal superhedging strategy for the buyer. The key to this is the following construction.

Construction 4.6. For all t , let

$$\mathcal{U}_t^b := \begin{cases} -\xi_t + \mathcal{S}_t & \text{on } \mathcal{E}_t, \\ \emptyset & \text{on } \Omega \setminus \mathcal{E}_t. \end{cases}$$

Define

$$\mathcal{V}_T^b := \mathcal{W}_T^b := \emptyset, \quad \mathcal{Z}_T^b := \mathcal{U}_T^b.$$

For $t < T$, let

$$\begin{aligned} \mathcal{W}_t^b &:= \mathcal{Z}_{t+1}^b \cap \mathcal{L}_t^d, \\ \mathcal{V}_t^b &:= \mathcal{W}_t^b + \mathcal{S}_t, \\ \mathcal{Z}_t^b &:= \mathcal{U}_t^b \cup \mathcal{V}_t^b. \end{aligned} \tag{4.9}$$

For each t the set \mathcal{U}_t^b is the collection of portfolios in \mathcal{L}_t that allows the buyer to be in a solvent position after exercising the option at time t . For $t < T$ the set \mathcal{W}_t^b is the collection of portfolios at time t that allow the buyer to superhedge the option in the future (at time $t+1$ or later) and \mathcal{V}_t^b consists of those portfolios that may be rebalanced at time t into a portfolio in \mathcal{W}_t^b . The set \mathcal{Z}_t^b consists of all portfolios that allow the buyer to exercise the option at time t or any time in the future.

Remark 4.7. Construction 4.6 differs from Construction 4.2 in two respects. Firstly, the payoff is treated differently because it is delivered by the seller and received by the buyer. Secondly, there is a union in (4.9) where there is an intersection in (4.5). This encapsulates the opposing positions of the seller and the buyer: any portfolio held by the seller at time t must enable him to settle the option at time t or later, whereas any portfolio held by the buyer needs to enable him to achieve solvency by exercising the option, either at time t or at some point in the future. The union in (4.9) also illustrates the fact that the pricing problem for the buyer is not convex.

Remark 4.8. On \mathcal{E}_t the set \mathcal{U}_t^b is polyhedral and non-empty. It is then possible to show the following by backward induction on t :

- $\mathcal{V}_t^b = \mathcal{W}_t^b = \mathcal{Z}_t^b = \emptyset$ on $\Omega \setminus \mathcal{E}_t^*$.
- $\mathcal{Z}_t^b = \mathcal{V}_t^b$ on $\Omega \setminus \mathcal{E}_t$ and $\mathcal{Z}_t^b = \mathcal{U}_t^b$ on $\Omega \setminus \mathcal{E}_{t+1}^*$.

- \mathcal{V}_t^b and \mathcal{W}_t^b on \mathcal{E}_{t+1}^* , and \mathcal{Z}_t^b on \mathcal{E}_t^* can be written as a finite union of non-empty closed polyhedral sets (but \mathcal{V}_t^b , \mathcal{W}_t^b and \mathcal{Z}_t^b are not convex in general).

Note in particular that since $\mathcal{E}_0^* = \Omega$, the last item applies to \mathcal{Z}_0^b , so it is non-empty and closed.

Here is the main pricing and hedging theorem for the buyer.

Theorem 4.9. The set \mathcal{Z}_0^b is the collection of initial endowments allowing superhedging of (ξ, \mathcal{E}) by the buyer, and

$$\begin{aligned}\pi_i^b(\xi, \mathcal{E}) &= \max_{\tau \in \mathcal{T}^\mathcal{E}} \min_{(\mathbb{P}, S) \in \mathcal{P}^i(\tau)} \mathbb{E}_{\mathbb{P}}((\xi \cdot S)_\tau) \\ &= \max_{\tau \in \mathcal{T}^\mathcal{E}} \inf_{(\mathbb{P}, S) \in \mathcal{P}^i(\tau)} \mathbb{E}_{\mathbb{P}}((\xi \cdot S)_\tau) \\ &= -\min\{x \in \mathbb{R} \mid xe^i \in \mathcal{Z}_0^b\}.\end{aligned}$$

An optimal superhedging strategy $(\check{y}, \check{\tau}) \in \Phi \times \mathcal{T}^\mathcal{E}$ with

$$\check{\tau} = \min\{t : \check{y}_t \in \mathcal{U}_t^b\}$$

can be constructed algorithmically, and so can a $\check{\tau}$ -approximate martingale pair $(\check{\mathbb{P}}, \check{S}) \in \bar{\mathcal{P}}^i(\check{\tau})$ such that

$$\mathbb{E}_{\check{\mathbb{P}}}((\xi \cdot \check{S})_{\check{\tau}}) = \pi_i^b(\xi, \mathcal{E}). \quad (4.10)$$

A $\check{\tau}$ -approximate martingale pair $(\check{\mathbb{P}}, \check{S})$ is called *optimal for the buyer* if it satisfies (4.10).

The proof of this theorem appears in Section 6, together with full details of the constructions therein.

4.3 Special cases

4.3.1 European options

Consider a European-style option that offers the payoff $\zeta \in \mathcal{L}_\tau$ at some given stopping time $\tau \in \mathcal{T}$ (in particular, we can have $\tau = T$ for an ordinary European option with expiry time T). Here \mathcal{L}_τ is the set of \mathbb{R}^d -valued \mathcal{F}_τ -measurable random variables. In our framework the payoff of such an option is the adapted process $\xi = (\xi_t)$ with

$$\xi_t = \zeta \mathbf{1}_{\{\tau=t\}} \text{ for all } t,$$

and its exercise policy $\mathcal{E} = (\mathcal{E}_t)$ is given by

$$\mathcal{E}_t := \{\tau = t\} \text{ for all } t.$$

It follows that $\mathcal{T}^\mathcal{E} = \{\tau\}$ and $\mathcal{T}^\mathcal{E} = \{\chi^\tau\}$. For clarity we denote this European option by (ζ, τ) instead of (ξ, \mathcal{E}) .

Observe that a trading strategy $y \in \Phi$ superhedges the option (ζ, τ) for the seller if and only if (y, τ) superhedges $(-\zeta, \tau)$ for the buyer. It also follows directly from (4.2) and (4.8) that

$$\pi_i^b(\zeta, \tau) = -\pi_i^a(-\zeta, \tau).$$

Thus the pricing and hedging problems for the buyer and seller of a European-style option are symmetrical. In particular, this means that the pricing problem for the buyer is convex, and the hedging problem for the seller does not involve any randomised stopping times.

Constructions 4.2 and 4.6 can be simplified considerably due to the simple structure of the exercise policy. Noting that at each time step t we have $\mathcal{U}_t^a = \xi_t + \mathcal{S}_t$ on $\{t = \tau\}$ and $\mathcal{U}_t^a = \mathbb{R}^d$ on $\{t \neq \tau\}$, Construction 4.2 can now be rewritten as follows for each t :

$$\begin{aligned} \mathcal{Z}_t^a &= \mathcal{V}_t^a = \mathcal{W}_t^a = \mathbb{R}^d && \text{on } \{t > \tau\}, \\ \mathcal{V}_t^a &= \mathcal{W}_t^a = \mathbb{R}^d && \text{on } \{t = \tau\}, \\ \mathcal{Z}_t^a &= \mathcal{U}_t^a = \xi_t + \mathcal{S}_t && \text{on } \{t = \tau\}, \\ \mathcal{Z}_t^a &= \mathcal{V}_t^a = \mathcal{W}_t + \mathcal{S}_t = \mathcal{Z}_{t+1} \cup \mathcal{L}_t && \text{on } \{t < \tau\}. \end{aligned} \quad (4.11)$$

$$(4.12)$$

Theorem 4.4 gives the ask price of (ζ, τ) as

$$\pi_i^a(\zeta, \tau) = \sup_{(\mathbb{P}, S) \in \mathcal{P}^i(\chi^\tau)} \mathbb{E}_{\mathbb{P}}((\xi \cdot S)_{\chi^\tau}) = \sup_{(\mathbb{P}, S) \in \mathcal{P}^i(\tau)} \mathbb{E}_{\mathbb{P}}(\zeta \cdot S_\tau) \quad (4.13)$$

since $\mathcal{P}(\tau) = \mathcal{P}(\chi^\tau)$ and $(\xi \cdot S)_{\chi^\tau} = (\xi \cdot S)_\tau = \zeta \cdot S_\tau$. A similar simplification is possible for the buyer; note that \mathcal{V}_t^b , \mathcal{W}_t^b and \mathcal{Z}_t^b are convex for all t . The bid price of (ζ, τ) is

$$\pi_i^b(\zeta, \tau) = -\pi_i^a(-\zeta, \tau) = \inf_{(\mathbb{P}, S) \in \mathcal{P}^i(\tau)} \mathbb{E}_{\mathbb{P}}(\zeta \cdot S_\tau),$$

which is consistent with Theorem 4.9.

Consider the special case $\tau = T$, which corresponds to a classical European option. The simplified construction (4.11)–(4.12) is the same as that of Löhne and Rudloff [13, Theorem 2]. The representations for the bid and ask prices can be simplified further by noting that

$$S_{t+1}^{\chi^T*} = S_T \text{ for all } t < T.$$

For any $(\mathbb{P}, S) \in \bar{\mathcal{P}}^i(\chi^T)$, the adapted process $\check{S} = (\check{S}_t)$ defined by

$$\check{S}_t := \mathbb{E}_{\mathbb{P}}(S_T | \mathcal{F}_t) \text{ for all } t$$

is a \mathbb{P} -martingale such that $(\mathbb{P}, \check{S}) \in \bar{\mathcal{P}}^i$, and

$$\mathbb{E}_{\mathbb{P}}(\zeta \cdot S_T) = \mathbb{E}_{\mathbb{P}}(\zeta \cdot \check{S}_T).$$

Thus the supremum in (4.13) need only be taken over \mathcal{P}^i , and it follows that

$$\pi_i^a(\zeta, T) = \sup_{(\mathbb{P}, S) \in \mathcal{P}^i} \mathbb{E}_{\mathbb{P}}(\zeta \cdot S_T) = \max_{(\mathbb{P}, S) \in \bar{\mathcal{P}}^i} \mathbb{E}_{\mathbb{P}}(\zeta \cdot S_T).$$

This result extends previous work by [1, 7, 20] in two-asset models. Its conclusions are technically closest to the non-constructive results for currency models in [6, 10].

4.3.2 Bermudan options

The exercise policy \mathcal{E} for a Bermudan option with payoff process ξ that can be exercised at given times $t_1 < \dots < t_n$ is defined in (3.1). The collections of ordinary and randomised stopping times consistent with this exercise policy are

$$\begin{aligned} \mathcal{T}^{\mathcal{E}} &= \{\tau \in \mathcal{T} : \tau \in \{t_1, \dots, t_n\}\}, \\ \mathcal{X}^{\mathcal{E}} &= \{\chi \in \mathcal{X} : \chi_t = 0 \text{ for all } t \notin \{t_1, \dots, t_n\}\}. \end{aligned}$$

Note that \mathcal{Z}_t^a and \mathcal{Z}_t^b are closed non-empty strict subsets of \mathcal{L}_t whenever $t \leq t_n$. Theorems 4.4 and 4.9 can then be used to compute $\pi_i^a(\xi, \mathcal{E})$ and $\pi_i^b(\xi, \mathcal{E})$. Moreover optimal superhedging strategies $y^a \in \Phi$ for the seller and $(y^b, \tau) \in \Phi \times \mathcal{T}^{\mathcal{E}}$ for the buyer can be constructed algorithmically.

4.3.3 American options

Consider an American-style option with random expiration date that offers the payoff ξ_{τ} at a stopping time $\tau \in \mathcal{T}$ chosen by the buyer, with the restriction that $\tau \leq \sigma$ for a given stopping time $\sigma \in \mathcal{T}$. The stopping time σ might be interpreted, for example, as the hitting time of a barrier.

The adapted process $\xi = (\xi_t)$ need only be defined up to time σ . The exercise policy $\mathcal{E} = (\mathcal{E}_t)$ satisfies $\mathcal{E}_t = \{\sigma \geq t\}$ for all t , and the sets of stopping times consistent with the exercise policy are

$$\begin{aligned} \mathcal{T} \wedge \sigma &:= \mathcal{T}^{\mathcal{E}} = \{\tau \in \mathcal{T} : \tau \leq \sigma\}, \\ \mathcal{X} \wedge \sigma &:= \mathcal{X}^{\mathcal{E}} = \{\chi \in \mathcal{X} : \chi_t = 0 \text{ on } \{t > \sigma\} \text{ for all } t\}. \end{aligned}$$

Denote this American-style option by (ξ, σ) instead of (ξ, \mathcal{E}) . Theorems 4.4 and 4.9 give the ask and bid prices as

$$\begin{aligned} \pi_i^a(\xi, \sigma) &= \max_{\chi \in \mathcal{X} \wedge \sigma} \sup_{(\mathbb{P}, S) \in \mathcal{P}^i(\chi)} \mathbb{E}_{\mathbb{P}}((\xi \cdot S)_{\chi}) = \max_{\chi \in \mathcal{X} \wedge \sigma} \max_{(\mathbb{P}, S) \in \bar{\mathcal{P}}^i(\chi)} \mathbb{E}_{\mathbb{P}}((\xi \cdot S)_{\chi}), \\ \pi_i^b(\xi, \sigma) &= \max_{\tau \in \mathcal{T} \wedge \sigma} \inf_{(\mathbb{P}, S) \in \mathcal{P}^i(\tau)} \mathbb{E}_{\mathbb{P}}((\xi \cdot S)_{\tau}) = \max_{\tau \in \mathcal{T} \wedge \sigma} \min_{(\mathbb{P}, S) \in \bar{\mathcal{P}}^i(\tau)} \mathbb{E}_{\mathbb{P}}((\xi \cdot S)_{\tau}). \end{aligned}$$

For each t the following simplifications hold in Constructions 4.2 and 4.6:

- $\mathcal{V}_t^a = \mathcal{W}_t^a = \mathbb{R}^d$ and $\mathcal{V}_t^b = \mathcal{W}_t^b = \emptyset$ on $\{t \geq \sigma\}$.
- $\mathcal{Z}_t^a = \mathcal{U}_t^a = \xi_t + \mathcal{S}_t$ and $\mathcal{Z}_t^b = \mathcal{U}_t^b = -\xi_t + \mathcal{S}_t$ on $\{t = \sigma\}$.

- $\mathcal{Z}_t^a = \mathcal{V}_t^a \cap (\xi_t + \mathcal{S}_t)$ and $\mathcal{Z}_t^b = \mathcal{V}_t^b \cup (-\xi_t + \mathcal{S}_t)$ on $\{t < \sigma\}$.

The special case $\sigma = T$ corresponds to a classical American option. In this case the bid and ask prices are

$$\begin{aligned}\pi_i^a(\xi, T) &= \max_{\chi \in \mathcal{X}} \sup_{(\mathbb{P}, S) \in \mathcal{P}^i(\chi)} \mathbb{E}_{\mathbb{P}}((\xi \cdot S)_{\chi}) = \max_{\chi \in \mathcal{X}} \max_{(\mathbb{P}, S) \in \bar{\mathcal{P}}^i(\chi)} \mathbb{E}_{\mathbb{P}}((\xi \cdot S)_{\chi}), \\ \pi_i^b(\xi, \sigma) &= \max_{\tau \in \mathcal{T}} \inf_{(\mathbb{P}, S) \in \mathcal{P}^i(\tau)} \mathbb{E}_{\mathbb{P}}((\xi \cdot S)_{\tau}) = \max_{\tau \in \mathcal{T}} \min_{(\mathbb{P}, S) \in \bar{\mathcal{P}}^i(\tau)} \mathbb{E}_{\mathbb{P}}((\xi \cdot S)_{\tau}).\end{aligned}$$

This directly extends previous results by [4, 21, 24] for two-asset models. In the context of currency models, this is consistent with the results of [2] for the seller.

Remark 4.10. In this work it is assumed that trading strategies are rebalanced at each time instant t only after it becomes known that the option is not to be exercised at that time instant. In their work [2] on pricing American options for the seller, Bouchard & Temam follow a different convention by assuming that the portfolios in a hedging strategy must be rebalanced before exercise decisions become known. The method in this paper also applies to their case, provided that the order of the operations in (4.4) and (4.5) is interchanged, i.e. replace these equations by

$$\mathcal{Z}_{t-1}^a := \mathcal{W}_{t-1}^a \cap \mathcal{U}_{t-1}^a + \mathcal{S}_{t-1}^a \text{ for } t > 0.$$

Ask prices obtained in this way are in general higher than the ask prices presented above. This is because a superhedging strategy for the seller in this setting will also superhedge under our definition, but the converse is not always true. Because of this, superhedging as we have defined above is easier to achieve, and it is therefore more natural for traders to follow than the approach of Bouchard & Temam.

Examples 5.6 and 6.2 below demonstrate the computation of the bid and ask prices of an American option in a toy model.

5 Pricing and hedging for the seller

This section is devoted to the proof of Theorem 4.4. Recall that a trading strategy $y \in \Phi$ superhedges the option (ξ, \mathcal{E}) for the seller if (4.1) holds. In view of Proposition 3.2, this is equivalent to

$$y_t - \xi_t \in \mathcal{S}_t \text{ on } \mathcal{E}_t \text{ for all } t$$

or

$$y_t \in \mathcal{U}_t^a \text{ for all } t. \tag{5.1}$$

We now have the following result.

Proposition 5.1. The ask price $\pi_i^a(\xi, \mathcal{E})$ defined in (4.2) is finite and

$$\pi_i^a(\xi, \mathcal{E}) \geq \sup_{\chi \in \mathcal{X}^\mathcal{E}} \sup_{(\mathbb{P}, S) \in \bar{\mathcal{P}}^i(\chi)} \mathbb{E}_\mathbb{P}((\xi \cdot S)_\chi) \quad (5.2)$$

$$= \sup_{\chi \in \mathcal{X}^\mathcal{E}} \sup_{(\mathbb{P}, S) \in \bar{\mathcal{P}}^i(\chi)} \mathbb{E}_\mathbb{P}((\xi \cdot S)_\chi). \quad (5.3)$$

Proof. We show by backward induction below that if $\chi \in \mathcal{X}^\mathcal{E}$, $(\mathbb{P}, S) \in \bar{\mathcal{P}}^i(\chi)$ and $y \in \Phi$ with $y_0 = xe^i$ superhedges (ξ, \mathcal{E}) for the seller, then

$$y_t \cdot \mathbb{E}_\mathbb{P}(S_t^{\chi*} | \mathcal{F}_t) \geq \mathbb{E}_\mathbb{P}((\xi \cdot S)_t^{\chi*} | \mathcal{F}_t) \quad (5.4)$$

for all t . The property $S^i \equiv 1$ then gives

$$x = xe^i \cdot \mathbb{E}_\mathbb{P}(S_\chi) = xe^i \cdot \mathbb{E}_\mathbb{P}(S_0^{\chi*} | \mathcal{F}_0) \geq \mathbb{E}_\mathbb{P}((\xi \cdot S)_0^{\chi*} | \mathcal{F}_0) = \mathbb{E}_\mathbb{P}((\xi \cdot S)_\chi),$$

and the inequality (5.2) is immediate. The equality (5.3) follows directly from Lemma 2.7. The property $-\infty < \pi_i^a(\xi, \mathcal{E}) < \infty$ holds true since $\bar{\mathcal{P}}^i(\chi) \neq \emptyset$ and (ξ, \mathcal{E}) has a trivial superhedging strategy for the seller, given by (y_t^1, \dots, y_t^d) where

$$y_t^j := \max\{\xi_s^{j\omega} : s = 0, \dots, T, \omega \in \mathcal{E}_s\} \quad (5.5)$$

for all j and t .

Observe that for any t we have $\chi_t = 0$ on $\Omega \setminus \mathcal{E}_t$, and on \mathcal{E}_t we have $y_t - \xi_t \in \mathcal{S}_t$, so that $y_t \cdot S_t \geq \xi_t \cdot S_t$ since $S_t \in \mathcal{S}_t^*$ (see Definition 2.6). This means that

$$\chi_t y_t \cdot S_t \geq \chi_t \xi_t \cdot S_t \text{ for all } t.$$

To prove (5.4) by backward induction, first note that at time T ,

$$y_T \cdot \mathbb{E}_\mathbb{P}(S_T^{\chi*} | \mathcal{F}_T) = \chi_T y_T \cdot S_T \geq \chi_T \xi_T \cdot S_T = \mathbb{E}_\mathbb{P}((\xi \cdot S)_T^{\chi*} | \mathcal{F}_T).$$

Suppose for some $t < T$ that

$$y_{t+1} \cdot \mathbb{E}_\mathbb{P}(S_{t+1}^{\chi*} | \mathcal{F}_{t+1}) \geq \mathbb{E}_\mathbb{P}((\xi \cdot S)_{t+1}^{\chi*} | \mathcal{F}_{t+1}).$$

The self-financing condition $y_t - y_{t+1} \in \mathcal{S}_t$ together with $\mathbb{E}_\mathbb{P}(S_{t+1}^{\chi*} | \mathcal{F}_t) \in \mathcal{S}_t^*$ (see Definition 2.6) gives

$$y_t \cdot \mathbb{E}_\mathbb{P}(S_{t+1}^{\chi*} | \mathcal{F}_t) \geq y_{t+1} \cdot \mathbb{E}_\mathbb{P}(S_{t+1}^{\chi*} | \mathcal{F}_t).$$

Combining this with the inductive assumption, we obtain

$$\begin{aligned} y_t \cdot \mathbb{E}_\mathbb{P}(S_t^{\chi*} | \mathcal{F}_t) &= \chi_t y_t \cdot S_t + y_t \cdot \mathbb{E}_\mathbb{P}(S_{t+1}^{\chi*} | \mathcal{F}_t) \\ &\geq \chi_t \xi_t \cdot S_t + y_{t+1} \cdot \mathbb{E}_\mathbb{P}(S_{t+1}^{\chi*} | \mathcal{F}_t) \\ &= \mathbb{E}_\mathbb{P}(\chi_t \xi_t \cdot S_t + y_{t+1} \cdot \mathbb{E}_\mathbb{P}(S_{t+1}^{\chi*} | \mathcal{F}_{t+1}) | \mathcal{F}_t) \\ &\geq \mathbb{E}_\mathbb{P}(\chi_t \xi_t \cdot S_t + \mathbb{E}_\mathbb{P}((\xi \cdot S)_{t+1}^{\chi*} | \mathcal{F}_{t+1}) | \mathcal{F}_t) \\ &= \mathbb{E}_\mathbb{P}(\chi_t \xi_t \cdot S_t + (\xi \cdot S)_{t+1}^{\chi*} | \mathcal{F}_t) \\ &= \mathbb{E}_\mathbb{P}((\xi \cdot S)_t^{\chi*} | \mathcal{F}_t). \end{aligned}$$

This concludes the inductive step. \square

The next result shows that Z_0^a is the set of initial endowments of self-financing trading strategies that allow the seller to superhedge (ξ, \mathcal{E}) . It also links Construction 4.2 with the problem of computing the ask price in (4.2).

Proposition 5.2. We have

$$Z_0^a = \{y_0 \in \mathbb{R}^d : y = (y_t) \in \Phi \text{ superhedges } (\xi, \mathcal{E}) \text{ for the seller}\}$$

and

$$\pi_i^a(\xi, \mathcal{E}) = \min\{x \in \mathbb{R} | xe^i \in Z_0^a\}. \quad (5.6)$$

Proof. We establish below that $y \in \Phi$ superhedges (ξ, \mathcal{E}) for the seller if and only if $y_t \in Z_t^a$ for all t . Equation (5.6) then follows directly from (4.2). The minimum in (5.6) is attained because Z_0^a is polyhedral, hence closed, and $\pi_i^a(\xi, \mathcal{E})$ is finite by Proposition 5.1.

If $y \in \Phi$ superhedges (ξ, \mathcal{E}) for the seller, then it satisfies (5.1), and clearly $y_T \in Z_T^a$. For any $t < T$ suppose inductively that $y_{t+1} \in Z_{t+1}^a$. We have $y_{t+1} \in \mathcal{W}_t^a$ since y is predictable, and $y_t \in \mathcal{V}_t^a$ since it is self-financing. Thus $y_t \in \mathcal{V}_t^a \cap \mathcal{U}_t^a = Z_t^a$, which concludes the inductive step.

For the converse, fix any $y_0 \in Z_0^a$. For any $t \geq 0$, suppose recursively that $y_t \in \mathcal{L}_{(t-1) \vee 0} \cap Z_t^a$ has already been constructed. Then $y_t \in \mathcal{V}_t^a = \mathcal{W}_t^a + \mathcal{S}_t$, so there exists $y_{t+1} \in \mathcal{W}_t^a = Z_{t+1}^a \cap \mathcal{L}_t$ such that $y_t - y_{t+1} \in \mathcal{S}_t$. The predictable process $y = (y_t)$ that we have constructed is self-financing and satisfies (5.1) since $Z_t^a \subseteq \mathcal{U}_t^a$ for all t . Thus it superhedges (ξ, \mathcal{E}) for the seller. \square

It is now easy to construct an optimal superhedging strategy $\hat{y} \in \Phi$ for the seller of (ξ, \mathcal{E}) by following the construction in the second half of the proof of Proposition 5.2 with $\hat{y}_0 := \pi_i^a(\xi, \mathcal{E})e^i \in Z_0^a$.

Consider now the following result.

Proposition 5.3. There exist $\hat{\chi} \in \mathcal{X}^\mathcal{E}$, $(\hat{\mathbb{P}}, \hat{S}) \in \bar{\mathcal{P}}^i(\hat{\chi})$ such that

$$\mathbb{E}_{\hat{\mathbb{P}}}((\xi \cdot \hat{S})_{\hat{\chi}}) = -\min\{Z_0^a(s) : s \in \sigma_i(\mathbb{R}^d)\}, \quad (5.7)$$

where Z_0^a is the support function of $-Z_0^a$.

With Proposition 5.3 in hand, the proof of Theorem 4.4 is straightforward, so we provide it now.

Theorem 4.4. Note from (5.6) that

$$\begin{aligned}
\pi_i^a(\xi, \mathcal{E}) &= \min\{x \in \mathbb{R} : xe^i \in \mathcal{Z}_0^a\} \\
&= \min\{x \in \mathbb{R} : -xe^i \in -\mathcal{Z}_0^a\} \\
&= \min\{x \in \mathbb{R} : -xe^i \cdot y \leq Z_0^a(y) \text{ for all } y \in \mathbb{R}^d\} \\
&\leq \inf\{x \in \mathbb{R} : -xe^i \cdot s \leq Z_0^a(s) \text{ for all } s \in \sigma_i(\mathbb{R}^d)\} \\
&= \inf\{x \in \mathbb{R} : -x \leq Z_0^a(s) \text{ for all } s \in \sigma_i(\mathbb{R}^d)\} \\
&= -\sup\{x \in \mathbb{R} : x \leq Z_0^a(s) \text{ for all } s \in \sigma_i(\mathbb{R}^d)\} \\
&= -\inf\{Z_0^a(s) : s \in \sigma_i(\mathbb{R}^d)\} \\
&= -\min\{Z_0^a(s) : s \in \sigma_i(\mathbb{R}^d)\} \\
&= \mathbb{E}_{\hat{\mathbb{P}}}((\xi \cdot \hat{S})_{\hat{\chi}}),
\end{aligned}$$

where the last equality follows from Proposition 5.3. Combining this with Proposition 5.1 completes the proof. \square

Note from the proof above that the randomised stopping time $\hat{\chi}$ and $\hat{\chi}$ -approximate martingale pair $(\hat{\mathbb{P}}, \hat{S})$ of Proposition 5.3 are optimal for the seller.

The remainder of this section is devoted to establishing Proposition 5.3. For all t , let U_t^a , V_t^a , W_t^a , Z_t^a be the support functions of $-\mathcal{U}_t^a$, $-\mathcal{V}_t^a$, $-\mathcal{W}_t^a$, $-\mathcal{Z}_t^a$, respectively.

Remark 5.4. Remark 4.3 gives $Z_t^a = V_t^a$ on $\Omega \setminus \mathcal{E}_t$ and $Z_t^a = U_t^a$ on $\Omega \setminus \mathcal{E}_{t+1}^*$ for all t . This means that $V_t^a = W_t^a = \delta_{\mathbb{R}^d}^*$ on $\Omega \setminus \mathcal{E}_{t+1}^*$, whence $Z_t^a = \delta_{\mathbb{R}^d}^*$ on $\Omega \setminus \mathcal{E}_t^*$ for all t . Moreover, these functions are all polyhedral whenever they are not equal to $\delta_{\mathbb{R}^d}^*$ [19, Corollary 19.2.1].

Proposition 5.3 depends on the following technical result.

Lemma 5.5.

(a) For all t and $y \in \mathcal{L}_t$ we have

$$U_t^a(y) = \begin{cases} -y \cdot \xi_t & \text{on } \{y \in \mathcal{S}_t^*\} \cap \mathcal{E}_t, \\ 0 & \text{on } \{y = 0\} \cap (\Omega \setminus \mathcal{E}_t), \\ \infty & \text{on } \{y \notin \mathcal{S}_t^*\} \cup [\{y \neq 0\} \cap (\Omega \setminus \mathcal{E}_t)], \end{cases} \quad (5.8)$$

$$V_t^a(y) = \begin{cases} W_t^a(y) & \text{on } \{y \in \mathcal{S}_t^*\}, \\ \infty & \text{on } \{y \notin \mathcal{S}_t^*\}. \end{cases} \quad (5.9)$$

(b) Fix any t and $\mu \in \Omega_t$.

(i) If $\mu \subseteq \mathcal{E}_t \cap \mathcal{E}_{t+1}^*$, then

$$Z_t^{a\mu} = \text{conv}\{U_t^{a\mu}, V_t^{a\mu}\} \quad (5.10)$$

and $\text{dom } Z_t^{a\mu} = \mathcal{S}_t^{*\mu}$. Moreover, for each $Y \in \sigma_i(\text{dom } Z_t^{a\mu})$ there exist $\lambda \in [0, 1]$, $X \in \sigma_i(\text{dom } V_t^{a\mu})$ and $S \in \sigma_i(\text{dom } U_t^{a\mu}) = \sigma_i(\mathcal{S}_t^{*\mu})$ such that

$$Z_t^{a\mu}(Y) = (1 - \lambda)V_t^{a\mu}(X) + \lambda U_t^{a\mu}(S), \quad Y = (1 - \lambda)X + \lambda S.$$

(ii) If $\mu \subseteq \mathcal{E}_t \setminus \mathcal{E}_{t+1}^*$, then

$$Z_t^{a\mu} = U_t^{a\mu}$$

and $\text{dom } Z_t^{a\mu} = \mathcal{S}_t^{*\mu}$.

(iii) If $\mu \subseteq \mathcal{E}_{t+1}^* \setminus \mathcal{E}_t$, then

$$Z_t^{a\mu} = V_t^{a\mu}$$

and $\text{dom } Z_t^{a\mu}$ is a compactly i -generated cone.

(iv) If $\mu \subseteq \Omega \setminus \mathcal{E}_t^*$, then

$$Z_t^{a\mu} = \delta_{\mathbb{R}^d}^*.$$

(c) For each $t < T$ and $\mu \in \Omega_t$ with $\mu \subseteq \mathcal{E}_{t+1}^*$, we have

$$W_t^{a\nu} = \text{conv}\{Z_{t+1}^{a\nu} : \nu \in \text{succ } \mu\} \quad (5.11)$$

and $\text{dom } W_t^{a\nu}$ is a compactly i -generated cone. Moreover, for every $X \in \sigma_i(\text{dom } W_t^{a\nu})$ there exist $p^\nu \geq 0$ and $Y^\nu \in \sigma_i(\text{dom } Z_{t+1}^{a\nu})$ for each $\nu \in \text{succ } \mu$ such that

$$W_t^{a\nu}(X) = \sum_{\nu \in \text{succ } \mu} p^\nu Z_{t+1}^{a\nu}(Y^\nu), \quad X = \sum_{\nu \in \text{succ } \mu} p^\nu Y^\nu, \quad 1 = \sum_{\nu \in \text{succ } \mu} p^\nu.$$

The proof of Lemma 5.5 is deferred to Appendix A.

Proposition 5.3. We construct the process $\hat{S} = (\hat{S}_t)$ by backward recursion, together with auxiliary adapted processes $\hat{X} = (\hat{X}_t)$, $\hat{Y} = (\hat{Y}_t)$, $\hat{\lambda} = (\hat{\lambda}_t)$ and predictable $\hat{p} = (\hat{p}_t)$. Fix any $(\mathbb{P}, S) \in \mathcal{P}^i$.

As $\mathcal{E}_0^* = \Omega$, Lemma 5.5(b) ensures that $\sigma_i(\text{dom } Z_0^a)$ is non-empty and compact, and there exists $\hat{Y}_0 \in \sigma_i(\text{dom } Z_0^a)$ such that

$$Z_0^a(\hat{Y}_0) = \min\{Z_0^a(s) : s \in \sigma_i(\mathbb{R}^d)\}.$$

Note that \hat{Y}_0 is an appropriate starting value for the recursion below since $\Omega \setminus \mathcal{E}_0^* = \emptyset$.

For any $t \geq 0$, suppose that \hat{Y}_t is an \mathcal{F}_t -measurable random variable such that $\hat{Y}_t \in \sigma^i(\text{dom } Z_t^a)$ on \mathcal{E}_t^* and $\hat{Y}_t = S_t$ on $\Omega \setminus \mathcal{E}_t^*$. For any $\mu \in \Omega_t$ we now construct $\hat{\lambda}_t^\mu \in [0, 1]$, $\hat{X}_t \in \sigma_i(\mathcal{S}_t^*)$ and $\hat{S}_t \in \sigma_i(\mathcal{S}_t^*)$ such that

$$\hat{Y}_t^\mu = (1 - \hat{\lambda}_t^\mu)\hat{X}_t^\mu + \hat{\lambda}_t^\mu \hat{S}_t^\mu. \quad (5.12)$$

There are four possibilities:

- If $\mu \subseteq \mathcal{E}_t \cap \mathcal{E}_{t+1}^*$, then Lemma 5.5(b)(i) ensures the existence of $\hat{\lambda}_t^\mu \in [0, 1]$, $\hat{X}_t^\mu \in \sigma_i(\text{dom } V_t^{a\mu})$ and $\hat{S}_t^\mu \in \sigma_i(\mathcal{S}_t^{*\mu})$ satisfying (5.12) and

$$Z_t^{a\mu}(\hat{Y}_t^\mu) = (1 - \hat{\lambda}_t^\mu) V_t^{a\mu}(\hat{X}_t^\mu) + \hat{\lambda}_t^\mu U_t^a(\hat{S}_t^\mu). \quad (5.13)$$

This possibility does not arise when $t = T$ because $\mathcal{E}_{T+1}^* = \emptyset$.

- If $\mu \subseteq \mathcal{E}_t \setminus \mathcal{E}_{t+1}^*$, then Lemma 5.5(b)(ii) applies. Choosing $\hat{\lambda}_t^\mu := 1$, $\hat{X}_t^\mu := S_t^\mu$ and $\hat{S}_t^\mu := \hat{Y}_t^\mu$ yields (5.12) and

$$Z_t^{a\mu}(\hat{Y}_t^\mu) = U_t^a(\hat{S}_t^\mu) = \hat{\lambda}_t^\mu U_t^a(\hat{S}_t^\mu). \quad (5.14)$$

- If $\mu \subseteq \mathcal{E}_{t+1}^* \setminus \mathcal{E}_t$, then Lemma 5.5(b)(iii) gives (5.12) and

$$Z_t^{a\mu}(\hat{Y}_t^\mu) = V_t^{a\mu}(\hat{X}_t^\mu) = (1 - \hat{\lambda}_t^\mu) V_t^{a\mu}(\hat{X}_t^\mu) \quad (5.15)$$

after defining $\hat{\lambda}_t^\mu := 0$, $\hat{X}_t^\mu := \hat{Y}_t^\mu$ and $\hat{S}_t^\mu := S_t^\mu$. This possibility does not arise when $t = T$ because $\mathcal{E}_{T+1}^* = \emptyset$.

- If $\mu \not\subseteq \mathcal{E}_t^*$, then $\hat{Y}_t^\mu = S_t^\mu$ by the recursive assumption, and

$$Z_t^{a\mu} = V_t^{a\mu} = U_t^{a\mu} = \delta_{\mathbb{R}^d}^*$$

by Remark 5.4 and Lemma 5.5(b)(iv). Defining $\lambda_t^\mu := 0$ and $\hat{S}_t^\mu := \hat{X}_t^\mu := S_t^\mu$ gives (5.12). This possibility does not arise when $t = 0$ because $\mathcal{E}_0^* = \Omega$.

Note that $\hat{X}_t \in \sigma_i(\text{dom } V_t^a)$ on \mathcal{E}_{t+1}^* and $\hat{X}_t = S_t$ on $\Omega \setminus \mathcal{E}_{t+1}^*$. For any $t < T$ and $\mu \in \Omega_t$ we now construct $(\hat{Y}_t^\nu)_{\nu \in \text{succ } \mu}$ and $(\hat{p}_{t+1}^\nu)_{\nu \in \text{succ } \mu}$ such that

$$1 = \sum_{\nu \in \text{succ } \mu} p_{t+1}^\nu, \quad (5.16)$$

$$\hat{X}_t^\mu = \sum_{\nu \in \text{succ } \mu} p_{t+1}^\nu \hat{Y}_{t+1}^\nu, \quad (5.17)$$

There are two possibilities:

- If $\mu \subseteq \mathcal{E}_{t+1}^*$, then $\hat{X}_t \in \sigma_i(\text{dom } V_t^a)$ and Lemma 5.5(c) assures the existence of $(\hat{Y}_t^\nu)_{\nu \in \text{succ } \mu}$ and $(\hat{p}_{t+1}^\nu)_{\nu \in \text{succ } \mu}$ with $\hat{p}_{t+1}^\nu \in [0, 1]$, $\hat{Y}_{t+1}^\nu \in \sigma_i(\text{dom } Z_{t+1}^{a\nu})$ for all $\nu \in \text{succ } \mu$ satisfying (5.16)–(5.17) and

$$V_t^{a\mu}(\hat{X}_t^\mu) = W_t^{a\nu}(\hat{X}_t^\mu) = \sum_{\nu \in \text{succ } \mu} \hat{p}_{t+1}^\nu Z_{t+1}^{a\nu}(\hat{Y}_{t+1}^\nu). \quad (5.18)$$

- If $\mu \not\subseteq \mathcal{E}_{t+1}^*$, then defining $\hat{Y}_{t+1}^\nu := S_{t+1}^\nu$ and $\hat{p}_{t+1}^\nu := \mathbb{P}(\nu|\mu)$ for all $\nu \in \text{succ } \mu$ gives (5.16)–(5.17).

This concludes the recursive step.

The probability measure $\hat{\mathbb{P}}$ is defined as

$$\hat{\mathbb{P}}(\omega) := \prod_{t=0}^{T-1} \hat{p}_{t+1}^\omega \text{ for all } \omega.$$

Then (5.16)–(5.18) gives

$$V_t^a(\hat{X}_t) = \mathbb{E}_{\hat{\mathbb{P}}}(Z_{t+1}^a(\hat{Y}_{t+1})|\mathcal{F}_t) \text{ on } \mathcal{E}_{t+1}^*, \quad \hat{X}_t = \mathbb{E}_{\hat{\mathbb{P}}}(\hat{Y}_{t+1}|\mathcal{F}_t) \quad (5.19)$$

for all $t < T$.

The randomised stopping time $\hat{\chi} = (\hat{\chi}_t)$ is defined by $\hat{\chi}_0 := \hat{\lambda}_0$ and

$$\hat{\chi}_t := \hat{\lambda}_t \left[1 - \sum_{s=0}^{t-1} \hat{\chi}_s \right]$$

for $t > 0$. It is clear from the construction that $\hat{\lambda}_t = 0$ on $\Omega \setminus \mathcal{E}_t$ for all t , which implies that $\hat{\chi} \in \mathcal{X}^\mathcal{E}$. Observe also that

$$\hat{\lambda}_t \hat{\chi}_t^* = \hat{\chi}_t, \quad (1 - \hat{\lambda}_t) \hat{\chi}_t^* = \hat{\chi}_t^* - \chi_t = \hat{\chi}_{t+1}^*$$

for all t ; recall that $\hat{\chi}_{T+1}^* = 0$ by definition. It follows from (5.12) that

$$\hat{\chi}_t^* \hat{Y}_t = \hat{\chi}_{t+1}^* \hat{X}_t + \hat{\chi}_t \hat{S}_t$$

for all t . Equations (5.8) and (5.14) give

$$\hat{\chi}_t^* Z_t^a(\hat{Y}_t) = \hat{\chi}_t U_t^a(\hat{S}_t) = -\hat{\chi}_t \xi_t \cdot \hat{S}_t \text{ on } \mathcal{E}_t \setminus \mathcal{E}_{t+1}^*. \quad (5.20)$$

Since $\hat{\chi}_t = 0$ on $\Omega \setminus \mathcal{E}_t$, equations (5.13) and (5.15) may be combined with (5.8) to yield

$$\hat{\chi}_t^* Z_t^a(\hat{Y}_t) = \hat{\chi}_{t+1}^* V_t^a(\hat{X}_t) - \hat{\chi}_t \xi_t \cdot \hat{S}_t \text{ on } \mathcal{E}_{t+1}^*. \quad (5.21)$$

It is possible to show by backward induction that

$$\mathbb{E}_{\hat{\mathbb{P}}}(\hat{S}_{t+1}^{\hat{\chi}^*}|\mathcal{F}_t) = \hat{\chi}_{t+1}^* \hat{X}_t \quad (5.22)$$

for all t . At time $t = T$ this follows from the notational conventions $\hat{S}_{T+1}^{\hat{\chi}^*} = 0$ and $\hat{\chi}_{T+1}^* = 0$. Suppose that (5.22) holds for some $t > 0$. Then

$$\begin{aligned} \mathbb{E}_{\hat{\mathbb{P}}}(\hat{S}_t^{\hat{\chi}^*}|\mathcal{F}_{t-1}) &= \mathbb{E}_{\hat{\mathbb{P}}}(\hat{\chi}_t \hat{S}_t + \hat{S}_{t+1}^{\hat{\chi}^*}|\mathcal{F}_{t-1}) \\ &= \mathbb{E}_{\hat{\mathbb{P}}}(\hat{\chi}_t \hat{S}_t + \mathbb{E}_{\hat{\mathbb{P}}}(\hat{S}_{t+1}^{\hat{\chi}^*}|\mathcal{F}_t)|\mathcal{F}_{t-1}) \\ &= \mathbb{E}_{\hat{\mathbb{P}}}(\hat{\chi}_t \hat{S}_t + \hat{\chi}_{t+1}^* \hat{X}_t|\mathcal{F}_{t-1}) \\ &= \hat{\chi}_t^* \mathbb{E}_{\hat{\mathbb{P}}}(\hat{Y}_t|\mathcal{F}_{t-1}) = \hat{\chi}_t^* \hat{X}_{t-1}, \end{aligned}$$

which concludes the inductive step.

We also show by backward induction that

$$\hat{\chi}_t^* Z_t^a(\hat{Y}_t) = -\mathbb{E}_{\hat{\mathbb{P}}}((\xi \cdot \hat{S})_{\hat{\chi}_t^*}^* | \mathcal{F}_t) \text{ on } \mathcal{E}_t^* \quad (5.23)$$

for all t . If $t = T$ then $\mathcal{E}_{T+1}^* = \emptyset$ and (5.20) gives

$$\hat{\chi}_T^* Z_T^a(\hat{Y}_T) = -\hat{\chi}_T \xi_T \cdot \hat{S}_T = -(\xi \cdot S)_T^{\hat{\chi}_T^*} = -\mathbb{E}_{\hat{\mathbb{P}}}((\xi \cdot S)_T^{\hat{\chi}_T^*} | \mathcal{F}_T)$$

on $\mathcal{E}_T = \mathcal{E}_T^*$. Suppose now that

$$\hat{\chi}_{t+1}^* Z_{t+1}^a(\hat{Y}_{t+1}) = -\mathbb{E}_{\hat{\mathbb{P}}}((\xi \cdot \hat{S})_{\hat{\chi}_{t+1}^*}^* | \mathcal{F}_{t+1}) \text{ on } \mathcal{E}_{t+1}^*$$

holds for some $t < T$. On the set $\mathcal{E}_t \setminus \mathcal{E}_{t+1}^*$ we have $\hat{\chi}_{t+1}^* = 0$, whence $(\xi \cdot S)_{\hat{\chi}_{t+1}^*}^* = 0$. Equation (5.20) then gives

$$\hat{\chi}_t^* Z_t^a(\hat{Y}_t) = -\hat{\chi}_t \xi_t \cdot \hat{S}_t - \mathbb{E}_{\hat{\mathbb{P}}}((\xi \cdot S)_{\hat{\chi}_{t+1}^*}^* | \mathcal{F}_t) = -\mathbb{E}_{\hat{\mathbb{P}}}((\xi \cdot S)_t^{\hat{\chi}_t^*} | \mathcal{F}_t).$$

On \mathcal{E}_{t+1}^* the equations (5.19) and (5.21) give

$$\begin{aligned} \hat{\chi}_t^* Z_t^a(\hat{Y}_t) &= \hat{\chi}_{t+1}^* V_t^a(\hat{X}_t) - \hat{\chi}_t \xi_t \cdot \hat{S}_t \\ &= \hat{\chi}_{t+1}^* \mathbb{E}_{\hat{\mathbb{P}}}(Z_{t+1}^a(\hat{Y}_{t+1}) | \mathcal{F}_t) - \hat{\chi}_t \xi_t \cdot \hat{S}_t \\ &= -\mathbb{E}_{\hat{\mathbb{P}}}((\xi \cdot \hat{S})_{\hat{\chi}_{t+1}^*}^* | \mathcal{F}_t) - \hat{\chi}_t \xi_t \cdot \hat{S}_t \\ &= -\mathbb{E}_{\hat{\mathbb{P}}}((\xi \cdot \hat{S})_t^{\hat{\chi}_t^*} | \mathcal{F}_t). \end{aligned}$$

This concludes the inductive step since $\mathcal{E}_t^* = [\mathcal{E}_t \setminus \mathcal{E}_{t+1}^*] \cup \mathcal{E}_{t+1}^*$.

To summarize, we have constructed a randomised stopping time $\hat{\chi} \in \mathcal{X}^{\mathcal{E}}$, a probability measure $\hat{\mathbb{P}}$ and an adapted process \hat{S} such that $\hat{S}_t \in \sigma_i(\mathcal{S}_t^*)$ and

$$\mathbb{E}_{\hat{\mathbb{P}}}(S_{t+1}^{\hat{\chi}_{t+1}^*} | \mathcal{F}_t) = \hat{\chi}_{t+1}^* \hat{X}_t \in \text{dom } V_t^a \subseteq \mathcal{S}_t^*$$

for all t . Equation (5.23) moreover gives

$$\mathbb{E}_{\hat{\mathbb{P}}}((\xi \cdot \hat{S})_{\hat{\chi}}) = -Z_0^a(\hat{Y}_0)$$

which leads to (5.7) and completes the proof of Proposition 5.3. \square

Example 5.6. Consider a single-step model with four nodes at time 1, that is, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, and three assets. We take asset 3 to be a cash account with zero interest rate, assume the cash prices of assets 1 and 2 in a friction-free market to be

	S_0^1	S_0^2	S_1^1	S_1^2
ω_1	10	20	8	18
ω_2	10	20	12	18
ω_3	10	20	8	22
ω_4	10	20	12	22

and introduce transaction costs at the rate $k = \frac{1}{6}$ in a similar manner as in Example 2.4, with the matrix-valued exchange rate process

$$\pi_t = \begin{bmatrix} 1 & (1+k)S_t^2/S_t^1 & (1+k)/S_t^1 \\ (1+k)S_t^1/S_t^2 & 1 & (1+k)/S_t^2 \\ (1+k)S_t^1 & (1+k)S_t^2 & 1 \end{bmatrix}$$

for $t = 0, 1$. Consider an American option with payoff process

	$\xi_0 = (\xi_0^1, \xi_0^2, \xi_0^3)$	$\xi_1 = (\xi_1^1, \xi_1^2, \xi_1^3)$
ω_1	$(1, -1, 33)$	$(-1, 1, 10)$
ω_2	$(1, -1, 33)$	$(-2, 1, 10)$
ω_3	$(1, -1, 33)$	$(-1, 2, 10)$
ω_4	$(1, -1, 33)$	$(-2, 2, 10)$

in this model; its exercise policy is $\mathcal{E}_0 = \mathcal{E}_1 = \Omega$.

Construction 4.2 is formulated in terms of the convex sets $\mathcal{U}_t^a, \mathcal{V}_t^a, \mathcal{W}_t^a, \mathcal{Z}_t^a$, but it is easier to visualise it by drawing the support functions $U_t^a, V_t^a, W_t^a, Z_t^a$ of $-\mathcal{U}_t^a, -\mathcal{V}_t^a, -\mathcal{W}_t^a, -\mathcal{Z}_t^a$ or indeed the sections $-\sigma_3(\text{epi } U_t^a), -\sigma_3(\text{epi } V_t^a), -\sigma_3(\text{epi } W_t^a), -\sigma_3(\text{epi } Z_t^a)$, which are shown in Figure 2 for the above single-step model with transaction costs. Observe that all the polyhedra in Figure 2 are unbounded below, but have been truncated when drawing the pictures.

The construction proceeds as follows:

- We start with $-\sigma_3(\text{epi } U_1^a) = -\sigma_3(\text{epi } Z_1^a)$ for all four nodes at time 1, represented by the four dark gray polyhedra in Figure 2(a). These are computed using (5.8) in Lemma 5.5.
- We then take the convex hull of these four polyhedra to obtain $-\sigma_3(\text{epi } W_0^a)$, the semi-transparent gray polyhedron in Figures 2(a), (b), (c), (d). Formula (5.11) in Lemma 5.5 is used here.
- Next, $-\sigma_3(\text{epi } V_0^a)$, the dark gray polyhedron in Figure 2(b), is the intersection of $-\sigma_3(\text{epi } W_0^a)$ and $-\sigma_3(\mathcal{S}_0^*)$, according to (5.9) in Lemma 5.5.
- Then we take $-\sigma_3(\text{epi } U_0^a)$, the dark gray polyhedron in Figure 2(c). This is computed using (5.8) in Lemma 5.5.
- Finally, we obtain $-\sigma_3(\text{epi } Z_0^a)$, the dark gray polyhedron in Figure 2(d), as the convex hull of $-\sigma_3(\text{epi } V_0^a)$ and $-\sigma_3(\text{epi } U_0^a)$, according to (5.10) in Lemma 5.5.

The ask price of the American option is the maximum of $-\sigma_3(\text{epi } Z_0^a)$; see Theorem 4.4. The polyhedron $-\sigma_3(\text{epi } Z_0^a)$ has 10 vertices:

$$\begin{aligned} & (10, 120/7, 181/7), (60/7, 132/7, 262/7), (35/3, 22, 106/3), (35/3, 70/3, 38), \\ & (60/7, 120/7, 184/7), (35/3, 20, 950/33), (11, 132/7, 194/7), (66/7, 22, 310/7), \\ & (60/7, 20, 3170/77), (10, 70/3, 134/3), \end{aligned}$$

and its highest point turns out to be at $\frac{134}{3} \cong 44.67$. This is the ask price of the American option.

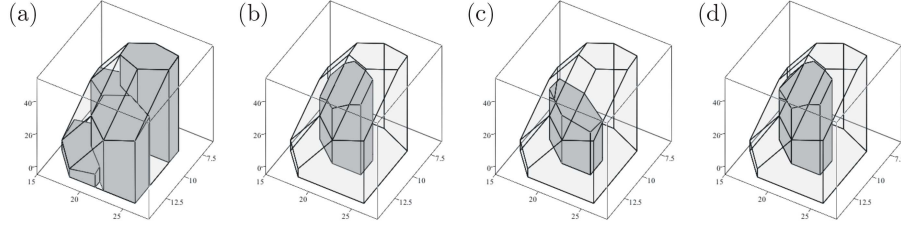


Figure 2: Construction 4.2 in the single-step model in Example 5.6 expressed in terms of $-\sigma_3(\text{epi } U_0^a)$, $-\sigma_3(\text{epi } V_0^a)$, $-\sigma_3(\text{epi } W_0^a)$, $-\sigma_3(\text{epi } Z_0^a)$

6 Pricing and hedging for the buyer

Recall that a pair (y, τ) consisting of a self-financing trading strategy $y \in \Phi$ and stopping time $\tau \in \mathcal{T}^\mathcal{E}$ superhedges the option (ξ, \mathcal{E}) for the buyer if (4.7) holds, equivalently if $y_\tau \in \mathcal{U}_\tau^b$.

The next result shows that the set \mathcal{Z}_0^b given by Construction 4.6 is the collection of initial endowments allowing the buyer to superhedge (ξ, \mathcal{E}) , and that it can be used to compute the bid price directly.

Proposition 6.1. We have

$$\begin{aligned} \mathcal{Z}_0^b &= \{y_0 \in \mathbb{R}^d : (y, \tau) \in \Phi \times \mathcal{T}^\mathcal{E} \text{ superhedges } (\xi, \mathcal{E}) \text{ for the buyer}\} \\ &= \{y_0 \in \mathbb{R}^d : (y, \tau) \in \Phi \times \mathcal{T}^\mathcal{E} \text{ superhedges } (\xi, \mathcal{E}) \text{ for the buyer} \\ &\quad \text{and } \tau = \min\{t : y_t \in \mathcal{U}_t^b\}\} \end{aligned}$$

and

$$\pi_i^b(\xi, \mathcal{E}) = -\min\{x \in \mathbb{R} \mid x e^i \in \mathcal{Z}_0^b\}. \quad (6.1)$$

Proof. We show below that $z \in \mathcal{Z}_0^b$ if and only if there exists a superhedging strategy (y, τ) for the buyer of (ξ, \mathcal{E}) with $y_0 = z$ and

$$\tau = \min\{t : y_t \in \mathcal{U}_t^b\}. \quad (6.2)$$

The two representations of \mathcal{Z}_0^b are equivalent since if (y, τ) superhedges (ξ, \mathcal{E}) for the buyer and

$$\tau' := \min\{t : y_t \in \mathcal{U}_t^b\},$$

then (y, τ') also superhedges (ξ, \mathcal{E}) for the buyer. Once the result for \mathcal{Z}_0^b is established, equation (6.1) follows directly from (4.8). The minimum is attained because \mathcal{Z}_0^b is closed and $\pi_i^b(\xi, \mathcal{E})$ is finite.

Suppose that $(y, \tau) \in \Phi \times \mathcal{T}^\mathcal{E}$ superhedges (ξ, \mathcal{E}) for the buyer and satisfies (6.2). We show by backward induction on t that $y_t \in \mathcal{Z}_t^b \setminus \mathcal{U}_t^b$ on $\{\tau > t\}$ for all t . At time $t = T$ this is trivial because $\{t > T\} = \emptyset$. For any $t < T$, suppose that $y_{t+1} \in \mathcal{Z}_{t+1}^b \setminus \mathcal{U}_{t+1}^b$ on $\{\tau > t+1\}$. Since $y_{t+1} \in \mathcal{U}_{t+1}^b$ on $\{\tau = t+1\}$,

this means that $y_{t+1} \in \mathcal{Z}_{t+1}^b$ on $\{\tau > t\} = \{\tau \geq t+1\}$. On $\{\tau > t\}$ we then have $y_{t+1} \in \mathcal{W}_t^b$ as $y_{t+1} \in \mathcal{L}_t$, and $y_t \in \mathcal{V}_t^b \subseteq \mathcal{Z}_t^b$ because of the self-financing property. However $y_t \notin \mathcal{U}_t^b$ on $\{\tau > t\}$ because of (6.2), and so $y_t \in \mathcal{Z}_t^b \setminus \mathcal{U}_t^b$ on $\{\tau > t\}$, which concludes the inductive step. Finally, $y_0 \in \mathcal{U}_0^b$ if $\tau = 0$ and $y_0 \in \mathcal{Z}_0^b \setminus \mathcal{U}_0^b$ if $\tau > 0$, and therefore $z := y_0 \in \mathcal{Z}_0^b$.

Conversely suppose that $z \in \mathcal{Z}_0^b$. We now construct a predictable process $y = (y_t)$ and a sequence (τ_t) of stopping times by recursion on t . At time 0, put

$$y_0 := z, \quad \tau_0 := \begin{cases} 0 & \text{if } y_0 \in \mathcal{U}_0^b, \\ 1 & \text{if } y_0 \in \mathcal{Z}_0^b \setminus \mathcal{U}_0^b. \end{cases}$$

For any $t > 0$ suppose that $y_{t-1} \in \mathcal{L}_{t-1}$ satisfies $y_{\tau_{t-1}} \in \mathcal{U}_{\tau_{t-1}}^b$ on $\{\tau_{t-1} < t\}$ and $y_{t-1} \in \mathcal{Z}_{t-1}^b \setminus \mathcal{U}_{t-1}^b$ on $\{\tau_{t-1} = t\}$, and that $\tau_{t-1} \in \mathcal{T}$ satisfies $\tau_{t-1} \leq t$ and $\{\tau_{t-1} = t\} \subseteq \mathcal{E}_t^*$. There are now two possibilities:

- On $\{\tau_{t-1} = t\}$ we have $y_{t-1} \in \mathcal{Z}_{t-1}^b \setminus \mathcal{U}_{t-1}^b$, which means that $y_{t-1} \in \mathcal{V}_{t-1}^b$, so there exists $y_t \in \mathcal{W}_{t-1}^b \subseteq \mathcal{Z}_t^b$ such that $y_{t-1} - y_t \in \mathcal{S}_{t-1}$.
- On $\{\tau_{t-1} < t\}$, put $y_t := y_{t-1}$ and note that $y_{t-1} - y_t = 0 \in \mathcal{S}_{t-1}$.

Also define

$$\tau_t := \begin{cases} \tau_{t-1} & \text{on } \{\tau_{t-1} < t\}, \\ t & \text{on } \{\tau_{t-1} = t\} \cap \{y_t \in \mathcal{U}_t^b\}, \\ t+1 & \text{on } \{\tau_{t-1} = t\} \cap \{y_t \in \mathcal{Z}_t^b \setminus \mathcal{U}_t^b\}. \end{cases}$$

Then $\tau_t \leq t+1$ and

$$\{\tau_t = t+1\} = \{\tau_{t-1} = t\} \cap \{y_t \in \mathcal{Z}_t^b \setminus \mathcal{U}_t^b\} \subseteq \mathcal{E}_t^* \cap \{\mathcal{V}_t^b \neq \emptyset\} = \mathcal{E}_t^* \cap \mathcal{E}_{t+1}^* = \mathcal{E}_{t+1}^*.$$

This concludes the recursive step.

Note that $\tau_T \in \mathcal{T}$ because $\mathcal{E}_{T+1}^* = \emptyset$ and moreover $\tau_T \in \mathcal{T}^\mathcal{E}$ since

$$\{\tau_T = t\} \subseteq \{y_t \in \mathcal{U}_t^b\} \subseteq \{\mathcal{U}_t^b \neq \emptyset\} = \mathcal{E}_t$$

for all t . Thus (y, τ_T) superhedges (ξ, \mathcal{E}) for the buyer and

$$\tau_T = \min\{t : y_t \in \mathcal{U}_t^b\}.$$

□

Let us now establish Theorem 4.9.

Theorem 4.9. Note first that (y, τ) is a superhedging strategy for the buyer of (ξ, \mathcal{E}) if and only if it is a superhedging strategy for the seller of the European-style option with payoff $-\xi_\tau$ and expiration date τ of Section 4.3.1. Denoting

the European-style option by $(-\xi_\tau, \tau)$, the bid price of (ξ, \mathcal{E}) defined in (4.8) can be written as

$$\begin{aligned}\pi_i^b(\xi, \mathcal{E}) &= \max_{\tau \in \mathcal{T}^\mathcal{E}} \sup\{-x \in \mathbb{R} : \exists y \in \Phi \text{ with } y_0 = xe^i \text{ such that } y_\tau + \xi_\tau \in \mathcal{S}_\tau\} \\ &= \max_{\tau \in \mathcal{T}^\mathcal{E}} [-\inf\{x \in \mathbb{R} : \exists y \in \Phi \text{ with } y_0 = xe^i \text{ such that } y_\tau + \xi_\tau \in \mathcal{S}_\tau\}] \\ &= \max_{\tau \in \mathcal{T}^\mathcal{E}} [-\pi_i^a(-\xi_\tau, \tau)].\end{aligned}\tag{6.3}$$

The equality (6.3) shows that $\pi_i^b(\xi, \mathcal{E})$ is finite because $\mathcal{T}^\mathcal{E}$ is finite and the ask prices are all finite by Proposition 5.1. Equation (4.13) in conjunction with Lemma 2.7 then gives

$$-\pi_i^a(-\xi_\tau, \tau) = \inf_{(\mathbb{P}, S) \in \mathcal{P}^i(\tau)} \mathbb{E}_\mathbb{P}((\xi \cdot S)_\tau) = \min_{(\mathbb{P}, S) \in \mathcal{P}^i(\tau)} \mathbb{E}_\mathbb{P}((\xi \cdot S)_\tau),\tag{6.4}$$

and so

$$\pi_i^b(\xi, \mathcal{E}) = \max_{\tau \in \mathcal{T}^\mathcal{E}} \inf_{(\mathbb{P}, S) \in \mathcal{P}^i(\tau)} \mathbb{E}_\mathbb{P}((\xi \cdot S)_\tau) = \max_{\tau \in \mathcal{T}^\mathcal{E}} \min_{(\mathbb{P}, S) \in \mathcal{P}^i(\tau)} \mathbb{E}_\mathbb{P}((\xi \cdot S)_\tau).$$

An optimal superhedging strategy $(\tilde{y}, \tilde{\tau})$ for the buyer of (ξ, \mathcal{E}) may be constructed using the second half of the proof of Proposition 6.1 below with $y_0 := z = -\pi_i^b(\xi, \mathcal{E})e^i$. Such a strategy $(\tilde{y}, \tilde{\tau})$ superhedges $(-\xi_{\tilde{\tau}}, \tilde{\tau})$ for the seller, so

$$\pi_i^b(\xi, \mathcal{E}) = -\tilde{y}_0^i \leq -\pi_i^a(-\xi_{\tilde{\tau}}, \tilde{\tau}) \leq \pi_i^b(\xi, \mathcal{E}),$$

whence

$$-\pi_i^a(-\xi, \mathcal{E}^{\tilde{\tau}}) = \pi_i^b(\xi, \mathcal{E}).\tag{6.5}$$

Thus the construction in the proof of Proposition 5.3 of the optimal stopping time and approximate martingale pair for the seller of the European option $(-\xi_{\tilde{\tau}}, \tilde{\tau})$ can be used to construct $\tilde{\chi}$ and $(\tilde{\mathbb{P}}, \tilde{S}) \in \tilde{\mathcal{P}}^i(\tilde{\chi})$ such that

$$\mathbb{E}_{\tilde{\mathbb{P}}}((-\xi \cdot \tilde{S})_{\tilde{\chi}}) = \pi_i^a(-\xi_{\tilde{\tau}}, \tilde{\tau}).$$

It is moreover clear from the construction in the proof of Proposition 5.3 and the structure of the exercise policy of $(-\xi_{\tilde{\tau}}, \tilde{\tau})$ that $\tilde{\chi} = \chi^{\tilde{\tau}}$. Thus $(\tilde{\mathbb{P}}, \tilde{S}) \in \tilde{\mathcal{P}}^i(\tilde{\tau})$ and

$$\mathbb{E}_{\tilde{\mathbb{P}}}((\xi \cdot \tilde{S})_{\tilde{\tau}}) = \mathbb{E}_{\tilde{\mathbb{P}}}((\xi \cdot \tilde{S})_{\tilde{\chi}}) = -\mathbb{E}_{\tilde{\mathbb{P}}}((-\xi \cdot \tilde{S})_{\tilde{\chi}}) = -\pi_i^a(-\xi_{\tilde{\tau}}, \tilde{\tau}) = \pi_i^b(\xi, \mathcal{E})$$

as required. \square

Example 6.2. Consider the computation of the bid price of the American option in Example 5.6 using Construction 4.6. In contrast to the seller's case, some of the sets $\mathcal{U}_t^b, \mathcal{V}_t^b, \mathcal{W}_t^b, \mathcal{Z}_t^b$ involved in this construction may fail to be convex, and there is no convex dual representation like that for the seller in Figure 2. To visualise the sets $\mathcal{U}_t^b, \mathcal{V}_t^b, \mathcal{W}_t^b, \mathcal{Z}_t^b$ we just draw their boundaries.

The construction for the buyer proceeds as follows:

- The first step is to compute $\mathcal{Z}_1^b = \mathcal{U}_1^b$ in each of the four scenarios; see Figure 3.
- Then we take the intersection of $\mathcal{U}_1^{b\omega_1}, \mathcal{U}_1^{b\omega_2}, \mathcal{U}_1^{b\omega_3}, \mathcal{U}_1^{b\omega_4}$ to obtain \mathcal{W}_0^b . This set appears in Figure 4(a).
- Next, the set \mathcal{V}_0^b in Figure 4(b) is the sum $\mathcal{W}_0^b + \mathcal{S}_0$ of \mathcal{W}_0^b and the solvency cone \mathcal{S}_0 .
- Then we take \mathcal{U}_0^b , which appears in Figure 4(c).
- Finally, the set \mathcal{Z}_0^b is the union of \mathcal{V}_0^b and \mathcal{U}_0^b . It appears in Figure 4(d); the dark gray region belongs to \mathcal{V}_0^b (but not \mathcal{U}_0^b), and the light gray region belongs to \mathcal{U}_0^b (but not \mathcal{V}_0^b).

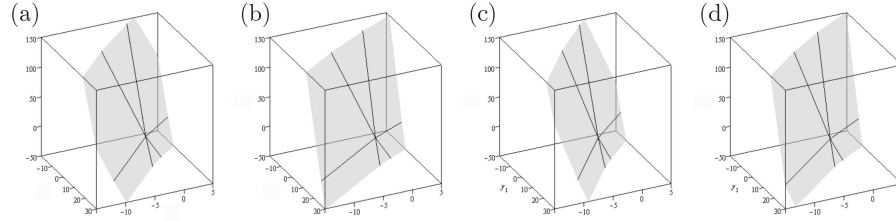


Figure 3: The sets $\mathcal{U}_1^{b\omega_1}, \mathcal{U}_1^{b\omega_2}, \mathcal{U}_1^{b\omega_3}, \mathcal{U}_1^{b\omega_4}$ in Construction 4.6 in Example 6.2

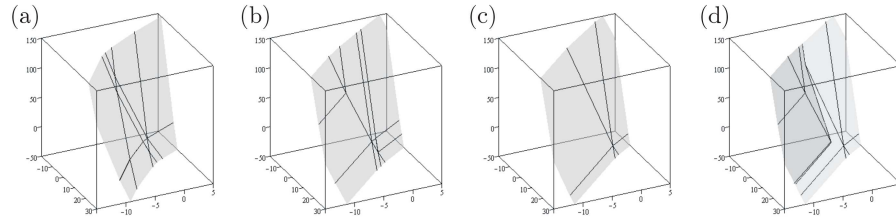


Figure 4: The sets $\mathcal{W}_0^b, \mathcal{V}_0^b, \mathcal{U}_0^b$ and \mathcal{Z}_0^b in Construction 4.6 in Example 6.2

The unbounded and non-convex set \mathcal{Z}_0^b has 8 vertices. Of these, the point $(-1, 1, -33)$ is a vertex of \mathcal{U}_0^b , the points $(4, -13/2, 163/2)$ and $(4, -15/7, -10)$ are vertices of \mathcal{V}_0^b , and

$$\begin{aligned} &(-1, -39/7, 361/3), (19/5, -15/7, -23/3), (39/10, -73/35, -10), \\ &(4, -233/112, -89/8), (127/30, -15/7, -12) \end{aligned}$$

are common to both \mathcal{U}_0^b and \mathcal{V}_0^b . The lowest number x such that $(0, 0, x) \in \mathcal{Z}_0^b$ is $x = -\frac{59}{3} \in \mathcal{U}_0^b$. By Theorem 4.9, the bid price of the option is $-x = \frac{59}{3} \cong 19.67$.

A Proof of Lemma 5.5

Lemma 5.5 in Section 5 depends on the following technical result.

Lemma A.1. Fix some $i = 1, \dots, d$, and let A_1, \dots, A_n be non-empty closed convex sets in \mathbb{R}^d such that $\text{dom } \delta_{A_k}^*$ is compactly i -generated for all k . Define $A := \bigcap_{k=1}^n A_k \neq \emptyset$; then

$$\delta_A^* = \text{conv}\{\delta_{A_1}^*, \dots, \delta_{A_n}^*\},$$

and for each $x \in \sigma_i(\text{dom } \delta_A^*)$ there exist $\alpha_1, \dots, \alpha_n \geq 0$ and x_1, \dots, x_n with $x_k \in \sigma_i(\text{dom } \delta_{A_k}^*)$ for all k such that

$$\delta_A^*(x) = \sum_{k=1}^n \alpha_k \delta_{A_k}^*(x_k), \quad \sum_{k=1}^n \alpha_k = 1, \quad \sum_{k=1}^n \alpha_k x_k = x.$$

The cone $\text{dom } \delta_A^*$ is moreover compactly i -generated and

$$\text{dom } \delta_A^* = \text{conv} \left[\bigcup_{k=1}^n \text{dom } \delta_{A_k}^* \right]. \quad (\text{A.1})$$

Proof. Let $f := \text{conv}\{\delta_{A_1}^*, \dots, \delta_{A_n}^*\}$. Then $\text{cl } f = \delta_A^*$ [19, Corollary 16.5.1], and since δ_A^* is proper it follows that f is proper and

$$\overline{\text{epi } f} = \text{epi } \delta_A^* \quad (\text{A.2})$$

by (2.1), so that $\delta_A^* = \text{cl } f \leq f$.

For any $k = 1, \dots, n$, the compact i -generation of $\text{dom } \delta_{A_k}^*$ means that $\sigma_i(\text{dom } \delta_{A_k}^*)$ is compact and non-empty. Thus the positive homogeneity of $\delta_{A_k}^*$ guarantees the existence of a closed proper convex function g_k with $\text{dom } g_k = \sigma_i(\text{dom } \delta_{A_k}^*)$ compact such that $\delta_{A_k}^*$ is *generated* by g_k , i.e.

$$\delta_{A_k}^*(y) = \begin{cases} \lambda g_k(x) & \text{if there exists } \lambda \geq 0 \text{ and } x \in \text{dom } g_k \text{ such that } y = \lambda x, \\ \infty & \text{otherwise.} \end{cases}$$

Let $g := \text{conv}\{g_1, \dots, g_n\}$; then

$$\text{dom } g = \text{conv} \left[\bigcup_{k=1}^n \sigma_i(\text{dom } \delta_{A_k}^*) \right]$$

is compact [19, Corollary 9.8.2]. Moreover, g is closed and proper, and for each $x \in \text{dom } g$ there exist $\alpha_1, \dots, \alpha_n \geq 0$ and x_1, \dots, x_n such that $x_k \in \sigma_i(\text{dom } \delta_{A_k}^*)$ for all k and

$$g(x) = \sum_{k=1}^n \alpha_k g_k(x_k), \quad \sum_{k=1}^n \alpha_k = 1, \quad \sum_{k=1}^n \alpha_k x_k = x \quad (\text{A.3})$$

[19, Corrolary 9.8.3; the common recession function is $\delta_{\mathbb{R}^d}^*$ since $\text{dom } g_k$ is compact for all k].

Let h be the positively homogeneous function generated by g , i.e.

$$h(y) := \begin{cases} \lambda g(x) & \text{if there exists } \lambda \geq 0 \text{ and } x \in \text{dom } g \text{ such that } y = \lambda x, \\ \infty & \text{otherwise.} \end{cases}$$

Clearly, h is a proper convex function and $\text{dom } h = \text{cone}(\text{dom } g)$ is compactly i -generated. The function h is moreover closed since

$$\text{epi } h = (\text{cone}(\text{epi } g)) \cup \{(0, \lambda) : \lambda \geq 0\} = \overline{\text{epi } h}$$

[19, Theorem 9.6], and it is majorised by $\delta_{A_1}^*, \dots, \delta_{A_n}^*$, hence $h \leq f$. Since h is closed, it then follows from (A.2) that

$$h \leq \delta_A^* \leq f. \quad (\text{A.4})$$

Fix any $y \in \text{dom } h$. There exist $\lambda \geq 0$ and $x \in \sigma_i(\text{dom } h) = \text{dom } g$ such that $y = \lambda x$. Fix any $\alpha_1, \dots, \alpha_n \geq 0$ and x_1, \dots, x_n satisfying (A.3) and where $x_k \in \sigma_i(\text{dom } \delta_{A_k}^*)$ for all k . Let $y_k := \lambda x_k$ for all k . Then

$$\sum_{k=1}^n \alpha_k y_k = \lambda \sum_{k=1}^n \alpha_k x_k = \lambda x = y$$

and

$$\sum_{k=1}^n \alpha_k \delta_{A_k}^*(y_k) = \lambda \sum_{k=1}^n \alpha_k g_k(x_k) = \lambda g(x) = h(y).$$

By definition of the convex hull, this means that $f(y) \leq h(y)$. Combining this with (A.4) gives

$$f = h = \delta_A^*.$$

The properties of $\text{dom } \delta_A^*$, in particular (A.1), then follow upon observing that

$$\text{dom } g = \sigma_i(\text{dom } h) = \sigma_i(\text{dom } \delta_A^*).$$

□

The paper concludes with the proof of Lemma 5.5.

Lemma 5.5. For each t , since \mathcal{S}_t is a cone, the support function of $-\mathcal{S}_t$ is

$$\delta_{-\mathcal{S}_t}^*(x) = \begin{cases} 0 & \text{if } x \cdot y \leq 0 \text{ for all } y \in -\mathcal{S}_t, \\ \infty & \text{otherwise} \end{cases} = \begin{cases} 0 & \text{if } x \in \mathcal{S}_t^*, \\ \infty & \text{otherwise.} \end{cases} \quad (\text{A.5})$$

Thus $\text{dom } \delta_{-\mathcal{S}_t}^* = \mathcal{S}_t^*$, and so $\text{dom } \delta_{-\mathcal{S}_t}^*$ is compactly i -generated.

For any t we have $U_t^a = \delta_{\mathbb{R}^d}^*$ on $\Omega \setminus \mathcal{E}_t$, together with

$$U_t^a(y) = \delta_{\{-\xi_t\}-\mathcal{S}_t}^*(y) = \delta_{\{-\xi_t\}}^*(y) + \delta_{-\mathcal{S}_t}^*(y) = -y \cdot \xi_t + \delta_{-\mathcal{S}_t}^*(y)$$

for $y \in \mathbb{R}^d$ on \mathcal{E}_t [19, p. 113]. Similarly,

$$V_t^a = \delta_{-\mathcal{W}_t^a - \mathcal{S}_t}^* = \delta_{-\mathcal{W}_t^a}^* + \delta_{-\mathcal{S}_t}^* = W_t^a + \delta_{-\mathcal{S}_t}^*.$$

Equalities (5.8) and (5.9) then follow from (2.2) and (A.5).

We now turn to Claims (b) and (c). Note first that the sets \mathcal{U}_t^a , \mathcal{V}_t^a , \mathcal{W}_t^a and \mathcal{Z}_t^a are non-empty for all t . This is easy to check by taking the trivial superhedging strategy for the seller defined by (5.5) and following the backward induction argument in the proof of Proposition 5.2.

We show below by backward induction that $\text{dom } Z_t^a$ is compactly i -generated on \mathcal{E}_t^* . While doing so we will establish Claims (b) and (c) for all t . At time T , using $Z_T^a = U_T^a$ and (4.3), the set $\text{dom } Z_T^a = \mathcal{S}_T^*$ is compactly i -generated on $\mathcal{E}_T^* = \mathcal{E}_T$, while $Z_T^a = \delta_{\mathbb{R}^d}^*$ on $\Omega \setminus \mathcal{E}_T^*$. This establishes Claim (b) for $t = T$ since $\mathcal{E}_{T+1}^* = \emptyset$.

At any time $t < T$, suppose that $\text{dom } Z_{t+1}^a$ is compactly i -generated on \mathcal{E}_{t+1}^* . For any $\mu \in \Omega_t$ there are now two possibilities:

- If $\mu \subseteq \mathcal{E}_{t+1}^*$, then Lemma A.1 applies to the sets $\{-Z_{t+1}^{a\nu} : \nu \in \text{succ } \mu\}$ since

$$\bigcap_{\nu \in \text{succ } \mu} Z_{t+1}^{a\nu} = \mathcal{W}_t^{a\mu} \neq \emptyset;$$

this immediately gives Claim (c). Moreover, the compact i -generation of $\text{dom } W_t^{a\mu}$ in combination with

$$\text{dom } V_t^{a\mu} = \text{dom } W_t^{a\mu} \cap \mathcal{S}_t^{*\mu}$$

shows that $\text{dom } V_t^{a\mu}$ is also compactly i -generated. There are now two possibilities:

- If $\mu \subseteq \mathcal{E}_t$, then Lemma A.1 applies to the sets $-\mathcal{U}_t^{a\mu}$ and $-\mathcal{V}_t^{a\mu}$. This gives Claim (b)(i) after noting that

$$\text{dom } Z_t^{a\mu} = \text{conv}(\text{dom } V_t^{a\mu} \cup \mathcal{S}_t^{*\mu}) = \mathcal{S}_t^{*\mu}$$

by (A.1).

- If $\mu \not\subseteq \mathcal{E}_t$, then $Z_t^{a\mu} = V_t^{a\mu}$ by Remark 5.4, which gives Claim (b)(iii).

- If $\mu \not\subseteq \mathcal{E}_{t+1}^*$, then $Z_t^{a\mu} = U_t^{a\mu}$ by Remark 5.4. There are again two possibilities:

- If $\mu \subseteq \mathcal{E}_t$, then (5.8) gives $\text{dom } Z_t^{a\mu} = \mathcal{S}_t^{*\mu}$. This is Claim (b)(ii).
- If $\mu \not\subseteq \mathcal{E}_t$, then (5.8) immediately gives Claim (b)(iv).

In summary, we have shown that $\text{dom } Z_t^a$ is compactly i -generated whenever

$$\mu \subseteq [\mathcal{E}_{t+1}^* \cap \mathcal{E}_t] \cup [\mathcal{E}_{t+1}^* \setminus \mathcal{E}_t] \cup [\mathcal{E}_t \setminus \mathcal{E}_{t+1}^*] = \mathcal{E}_t^*.$$

This concludes the inductive step, and completes the proof of Lemma 5.5. \square

References

- [1] Bensaid, B., Lesne, J.P., Pagès, H., Scheinkman, J.: Derivative asset pricing with transaction costs. *Mathematical Finance* **2**, 63–86 (1992)
- [2] Bouchard, B., Temam, E.: On the hedging of American options in discrete time markets with proportional transaction costs. *Electronic Journal of Probability* **10**, 746–760 (2005)
- [3] Boyle, P.P., Vorst, T.: Option replication in discrete time with transaction costs. *The Journal of Finance* **XLVII**(1), 347–382 (1992)
- [4] Chalasani, P., Jha, S.: Randomized stopping times and American option pricing with transaction costs. *Mathematical Finance* **11**(1), 33–77 (2001)
- [5] Chen, G.Y., Palmer, K., Sheu, Y.C.: The least cost super replicating portfolio in the Boyle-Vorst model with transaction costs. *International Journal of Theoretical and Applied Finance* **11**(1), 55–85 (2008)
- [6] Delbaen, F., Kabanov, Y.M., Valkeila, E.: Hedging under transaction costs in currency markets: A discrete-time model. *Mathematical Finance* **12**, 45–61 (2002)
- [7] Edirisinghe, C., Naik, V., Uppal, R.: Optimal replication of options with transactions costs and trading restrictions. *The Journal of Financial and Quantitative Analysis* **28**(1), 117–138 (1993)
- [8] Kabanov, Y.M.: Hedging and liquidation under transaction costs in currency markets. *Finance and Stochastics* **3**, 237–248 (1999)
- [9] Kabanov, Y.M., Rásonyi, M., Stricker, C.: No-arbitrage criteria for financial markets with efficient friction. *Finance and Stochastics* **6**, 371–382 (2002)
- [10] Kabanov, Y.M., Stricker, C.: The Harrison-Pliska arbitrage pricing theorem under transaction costs. *Journal of Mathematical Economics* **35**, 185–196 (2001)
- [11] Kociński, M.: Optimality of the replicating strategy for American options. *Applicationes Mathematicae* **26**(1), 93–105 (1999)
- [12] Kociński, M.: Pricing of the American option in discrete time under proportional transaction costs. *Mathematical Methods of Operations Research* **53**, 67–88 (2001)
- [13] Löhne, A., Rudloff, B.: An algorithm for calculating the set of superhedging portfolios and strategies in markets with transaction costs (2011). URL http://arxiv.org/PS_cache/arxiv/pdf/1107/1107.5720v1.pdf
- [14] Palmer, K.: A note on the Boyle-Vorst discrete-time option pricing model with transactions costs. *Mathematical Finance* **11**(3), 357–363 (2001)

- [15] Pennanen, T., King, A.J.: Arbitrage pricing of American contingent claims in incomplete markets - a convex optimization approach. Stochastic Programming E-Print Series 14 (2004). URL <http://edoc.hu-berlin.de/docviews/abstract.php?id=26772>
- [16] Perrakis, S., Lefoll, J.: Derivative asset pricing with transaction costs: An extension. Computational Economics **10**, 359–376 (1997)
- [17] Perrakis, S., Lefoll, J.: Option pricing and replication with transaction costs and dividends. Journal of Economic Dynamics and Control **24**, 1527–1561 (2000)
- [18] Perrakis, S., Lefoll, J.: The American put under transactions costs. Journal of Economic Dynamics and Control **28**, 915–935 (2004)
- [19] Rockafellar, R.T.: Convex Analysis. Princeton Landmarks in Mathematics and Physics. Princeton University Press (1996)
- [20] Roux, A., Tokarz, K., Zastawniak, T.: Options under proportional transaction costs: An algorithmic approach to pricing and hedging. Acta Applicandae Mathematicae **103**(2), 201–219 (2008). DOI 10.1007/s10440-008-9231-5
- [21] Roux, A., Zastawniak, T.: American options under proportional transaction costs: Pricing, hedging and stopping algorithms for long and short positions. Acta Applicandae Mathematicae **106**, 199–228 (2009). DOI 10.1007/s10440-008-9290-7
- [22] Rutkowski, M.: Optimality of replication in the CRR model with transaction costs. Applicationes Mathematicae **25**(1), 29–53 (1998)
- [23] Schachermayer, W.: The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time. Mathematical Finance **14**(1), 19–48 (2004)
- [24] Tokarz, K., Zastawniak, T.: American contingent claims under small proportional transaction costs. Journal of Mathematical Economics **43**(1), 65–85 (2006)